Math 170E
Week 3

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1 Conditional Probability and Bayes' Theorem

Example 1. (PSI 1.2.7) In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select

1. 6, 7, 8, 9.

   (A) $\frac{24}{10^4}$ (B) $\frac{24}{10\cdot9\cdot8\cdot7}$ (C) $\frac{4}{10\cdot9\cdot8\cdot7}$ (D) $\frac{12}{10^4}$ (E) $\frac{4^4}{10^4}$

   $\frac{4!}{10^4}$

2. 6, 7, 8, 8.

   (A) $\frac{24}{10^4}$ (B) $\frac{24}{10\cdot9\cdot8\cdot7}$ (C) $\frac{4}{10\cdot9\cdot8\cdot7}$ (D) $\frac{12}{10^4}$ (E) $\frac{4^4}{10^4}$

   $\frac{\frac{4!}{2!}}{10^4}$

3. 7, 7, 8, 8.

   (A) $\frac{6}{10^4}$ (B) $\frac{6}{10\cdot9\cdot8\cdot7}$ (C) $\frac{4}{10\cdot9\cdot8\cdot7}$ (D) $\frac{12}{10^4}$ (E) $\frac{24}{10^4}$

   $\frac{\frac{4!}{2!\cdot2!}}{10^4}$

4. 7, 8, 8, 8.

   (A) $\frac{4}{10^4}$ (B) $\frac{24}{10\cdot9\cdot8\cdot7}$ (C) $\frac{4}{10\cdot9\cdot8\cdot7}$ (D) $\frac{12}{10^4}$ (E) $\frac{4^4}{10^4}$

   $\frac{\frac{4!}{3!}}{10^4}$

Example 2. Monica throws two dice in a backgammon game. You know that the sum of two dice is 10. What is the probability that one of the dice is 5?

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

Example 3. (PSI 1.3.7) An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{5}$ (D) $\frac{1}{6}$ (E) $\frac{2}{5}$

Example 4. (PSI 1.5.9 - modified) There is a new diagnostic test for *coronavirus* that occurs in about 10% of the population. The test is not perfect, but will detect a person with the disease 95% of the time. It will, however, say that a person without the disease has the disease about 20% of the time. A person is selected at random from the population, and the test indicates that this person has the disease. What is the conditional probabilities that the person actually has the disease?

(A) $\frac{1}{3}$ (B) $\frac{9}{100}$ (C) $\frac{19}{55}$ (D) $\frac{15}{66}$ (E) $\frac{33}{95}$

Example 5. For two event A and B in the probability space, $P(A \cup B) = 0.6, P(A) = 0.2, P(B) = 0.5$. Determine if two events are independent.

Yes.

Example 6. For each of the following, determine the constant c so that $f(x)$ satisfies the conditions of being a pmf for a random variable X, and then depict each pmf as a line graph:

1. $f(x) = x/c,\ x = 1, 2, 3, 4.$ (A)1 (B)3 (C)4 (D)6 (E)10
2. $f(x) = cx,\ x = 1, 2, 3, ..., 10.$ (A) $\frac{1}{3}$ (B) $\frac{1}{10}$ (C) $\frac{1}{45}$ (D) $\frac{1}{55}$ (E) $\frac{1}{100}$
3. $f(x) = c(\frac{1}{4})^x,\ x = 1, 2, 3, \ldots$ (A)3 (B) $\frac{1}{3}$ (C)4 (D) $\frac{1}{4}$ (E)1
4. $f(x) = c(x + 1)^2, x = 0, 1, 2, 3.$ → $\frac{1}{30}$
5. $f(x) = \frac{c}{(x)(x+1)},\ x = 2, 3, 4, \ldots$
   Hint: Write $f(x) = c/x - c/(x + 1)$.
   → 2

Example 7. (Old Midterm Problem)

① $\frac{45 \cdot 3^8}{10^{10}}$

② $\frac{45 \cdot 2^{14}}{10^{10}}$

A hospital receives 70% of its flu vaccine from Company Good and the remainder from Company Evil. Each shipment contains a large number of vials of vaccine. From Company Good, 10% of the vials are ineffective. From Company Evil, 80% are ineffective. A hospital tests $n = 10$ randomly selected vials from one shipment for their effectiveness.

1. Compute the probability that exactly two of these 10 vials are ineffective if this shipment comes from Company Good. (You do not need to simplify)
2. Compute the probability that exactly two of these 10 vials are ineffective if this shipment comes from Company Evil. (You do not need to simplify)
3. Compute the probability that exactly two of these 10 vials are ineffective. (You do not need to simplify.)
4. Compute the conditional probability that this shipment come from Company Good. (Hint: Bayes theorem might be useful here) (You do not need to simplify.)

Example 8. Old Quiz Problem Let a random experiment be the casting of a pair of fair six-sided dice and let X equal the minimum of the two outcomes.

1. With reasonable assumptions, find the pmf of X. $f(6) = \frac{1}{36}, f(5) = \frac{3}{36}, f(4) = \frac{5}{36}, \ldots, f(1) = \frac{11}{36}$
2. Compute the mean of X, $E[X]$. → $\frac{11 \cdot 1 + 9 \cdot 2 + 7 \cdot 3 + 5 \cdot 4 + 3 \cdot 5 + 1 \cdot 6}{36} = \frac{91}{36}$
3. Compute $E[2X + 1]$.
   → $\frac{109}{18}$

→ ③ $\frac{45 \cdot 3^8 \cdot 7 + 45 \cdot 2^{14} \cdot 3}{10^{11}}$

④ $\frac{7 \cdot 3^8}{7 \cdot 3^8 + 3 \cdot 2^{14}}$

Example 9. (PSI 1.3.9) An urn contains four balls numbered 1 through 4. The balls are selected one at a time without replacement. A match occurs if the ball numbered m is the m^{th} ball selected. Let the event A_i denote a match on the i^{th} draw, for $i = 1, 2, 3, 4$.

$P(A_i) = \frac{3!}{4!} = \frac{1}{4}$

1. Find $P(A_i)$ for each i.

   (A)1 (B)$\frac{1}{2}$ (C)$\frac{1}{3}$ (D)$\frac{1}{4}$ (E)$\frac{1}{24}$

2. Find $P(A_i \cap A_j)$ for each $i \neq j$.

   (A)$\frac{1}{4}$ (B)$\frac{1}{6}$ (C)$\frac{1}{12}$ (D)$\frac{1}{24}$ (E)$\frac{1}{120}$

   $P(A_i \cap A_j) = \frac{2!}{4!} = \frac{1}{12}$

3. Find $P(A_i \cap A_j \cap A_k)$ for each $i \neq j \neq k \neq i$.

   (A)$\frac{1}{4}$ (B)$\frac{1}{6}$ (C)$\frac{1}{12}$ (D)$\frac{1}{24}$ (E)$\frac{1}{120}$

4. Compute $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Note that this represent the probability that there is at least one match.

   HINT: Use principle of inclusion & exclusion

   (A)$\frac{1}{4}$ (B)$\frac{2}{3}$ (C)$\frac{3}{5}$ (D)$\frac{5}{8}$ (E)$\frac{17}{24}$

5. Extend this exercise so that there are n balls in the urn. Show that the probability of at least one match is

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^{n+1}}{n!}$$

PIE example

6. What is the limit of this probability as n increases without bound?

$$= 1 - \left(\underbrace{1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots}_{} \right)$$

Taylor expansion of e^x when $x = -1$.

$$= 1 - \frac{1}{e}$$