Math 170E Notes

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End:

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-1 First Day: Introduction and Logistics

-1.1 Logistics

1. Instructor:
2. Teaching Assistant: Osman Akar
3. **Discussion Time**:
4. **Office Hours:** You are most welcomed to attend the office hours, even though you do not have questions but you want to see other student's questions, or you just want to chat in general. I was also UCLA undergrad and I have spent my 5 years here, so I have quite an experience of being a math major.
   - I understand that many of you are in different time zones. So if you cannot make any of my OHs, you can schedule an OH with me instead. I will try to be as accommodating as possible. Note that in this case I will email the whole class to notify that I will be holding an additional OH.
5. Email: oak@math.ucla.edu.
6. Email Policy: Please do not send me math questions via email, instead come to my office hours. It is quite time consuming to answer math questions via email.
7. Lecture notes: I will upload the written notes to CCLE under the corresponding week.
8. Recorded lecture videos: I will record and upload the lecture video to CCLE, **unless there is a significant drop in attendance**. I do not want to lecture to an empty class, and our discussions will be interactive with question pools.
9. Useful Links (Free materials UCLA offers):
   (a) MATLAB is free for UCLA Students: https://softwarecentral.ucla.edu/matlab-getmatlab
   (b) Microsoft Office is also free: https://www.it.ucla.edu/news/microsoft-office-proplus
   (c) You can read *The Economist* magazine for free if you are on UCLA network. If not, you can use UCLA VPN. See here for more info https://www.anderson.ucla.edu/rosenfeld-library/databases/business-databases-by-name/economist. Note that being on UCLA network is also useful to download articles from a number of publishers and also from UCLA library (NOTE: for some reason this free access does not seem to be working for a couple of weeks, but I hope they will fix it soon).

0 Ch0: Warm-Up Problems

1. A casino offers the following game. They give you a four sided fair dice. First you roll the first dice, then they pay you the number of dollars that you roll. If the cost of this game is 3$, would you play it? What is your expected loss/profit?

2. What if the cost was 2$. Would you change your answer?

3. The same casino offers the following game as well. They give you 2 four sided fair dice. First you roll the first dice, say that outcome is D_1. Then you roll the second dice D_1 times, and add up the all outcomes. Say this is P. Then they pay you P dollars. For example, you roll 3 in the first dice, so that $D_1 = 3$. Then you roll the second dice 3 times, say the outcomes are $4, 2, 4$. Then they pay you $P = 4 + 2 + 4 = 10$$. If the cost of this game is 6$, would you play it? What is your expected loss/profit?

4. You are given an unfair coin, so that $P(Head) = \frac{1}{3}$. You throw your coin until you hit tails.
   (a) Compute the probability that you only roll twice.
   (b) Compute the probability that you roll at most four times.
   (c) Compute the probability that you roll at most seven times.

5. You are given an unfair coin, so that $P(Head) = p \in (0, 1)$ is unknown. Your goal is to approximate p. How can you do it?

1 Ch1: Probability (An Introduction)

1.1 Ch 1.1: Properties of Probability and Introduction: Events as Sets

What is *Probability*: Probability is a method to quantify the possibility of occurrence of events. Let's start with an example: assume we have a class of 50 students of boys and girls, some have green eyes and the others have blue eyes.

Class		
	Boys	Girls
Blue	5	10
Green	15	20

The teacher chooses one student uniform at random in the class. Say that student is S. Let's define two events $A = \{S \text{ is a boy}\},\ B = \{S \text{ has green eyes}\}$. The sets A and B can be represented as yellow and red colored areas

	Boys	Girls
Blue	5	10
Green	15	20
A		

	Boys	Girls
Blue	5	10
Green	15	20
B		

S is chosen uniformly random, meaning that each student in the class have the same chance to be chosen, thus

$$P(\text{S is a boy}) = p(A) = \frac{\text{Number of Boys in the Class}}{\text{Total Number of Students}} = \frac{20}{50}$$

similarly

$$p(B) = \frac{\text{Number of Green Eyed Students in the Class}}{\text{Total Number of Students}} = \frac{35}{50}$$

$$p(A \cap B) = \frac{\text{Number of Green Eyed Boys in the Class}}{\text{Total Number of Students}} = \frac{15}{50}$$

$$p(A \cup B) = \frac{\text{Number of students either are boys or have Green Eyes in the Class}}{\text{Total Number of Students}} = \frac{40}{50}$$

Note that in this example, we defined the experiment (teacher choosing one student at random), and some the event (chosen student being a male or having green eyes), then question what is the possibility/probability of these particular events are happening.

Definition 1.1. Set Operations and Their Meaning in Probability As you see in the above example, we represent events as sets. This means that we have a universal set of all possibilities, and we represent event with particular conditions as a set in the universal set. E.g., we defined $A = \{S \text{ is a boy}\},\ B = \{S \text{ has green eyes}\}$.

$$\begin{aligned}
A \cup B &= (A \text{ union } B) = (A \text{ or } B) = (S \text{ is either boy or has green eyes}) \\
A \cap B &= (A \text{ intersection } B) = (A \text{ and } B) = (S \text{ is boy and has green eyes}) \\
\mathbb{P}(A \cup B) &= \text{Probability that } A \text{ or } B \text{ happens} \\
\mathbb{P}(A \cap B) &= \text{Probability that } A \text{ and } B \text{ happens} \\
A' &= A^c = (\text{Complement of } A) = (S \text{ is not a boy}) \\
\mathbb{P}(A') &= 1 - \mathbb{P}(A)
\end{aligned}$$

Example 1.1. (PSI 1.1.7) Given that $\mathbb{P}(A \cup B) = 0.76$ and $\mathbb{P}(A \cup B') = 0.87$, find $\mathbb{P}(A)$.

Solution We can use directly the set operations. In doing so, a **Venn Diagram** is the most helpful.

Theorem 1.1 (Principle of Inclusion & Exclusion (PIE)). *Let A, B, C be sets in probability space $(\mathbb{P}, \Omega)$. Then*

1. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$
2. $\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + P(C) - \mathbb{P}P(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$

Note: *It is a good example to prove above equalities*

Example 1.2. At UCLA 70 percent of students can speak either French or Spanish respectively. If half of the UCLA students can speak Spanish and 30 percent of UCLA students can speak French, what percentage of students can speak both.

Solution

Problem Setup By F and S, let's define the sets of UCLA students who can speak French and Spanish. The we are given the following:

- *In UCLA 70 percent of students can speak either French or Spanish.* $\Rightarrow \mathbb{P}(F \cup S) = 0.7$
- *Half of the UCLA students can speak Spanish* $\Rightarrow \mathbb{P}(S) = 0.5$
- *30 percent of UCLA students can speak French* $\Rightarrow \mathbb{P}(F) = 0.3$
- **Question:** *What percentage of students can speak both?* $\Rightarrow \mathbb{P}(S \cap F) =?$

Solution 1 We can use PIE. We have $\mathbb{P}(F \cup S) = \mathbb{F} + \mathbb{S} - \mathbb{P}(F \cap S)$. Hence $\mathbb{P}(F \cap S) = \mathbb{F} + \mathbb{S} - \mathbb{P}(F \cup S) = 0.3 + 0.5 - 0.7 = 0.1$, meaning 10 percent of the students can speak both of the languages.

Solution We can use directly the set operations. In doing so, a **Venn Diagram** is the most helpful.

Example 1.3. (PSI 1.1.1) Of a group of patients having injuries, 28% visit both a physical therapist and a chiropractor and 8% visit neither. Say that the probability of visiting a physical therapist exceeds the probability of visiting a chiropractor by 16%. What is the probability of a randomly selected person from this group visiting a physical therapist?

Solution Problem Setup Let S be the randomly selected person. Define 2 events:

$$A = \{\text{S visits physical therapist (PT)}\}$$

and

$$B = \{\text{S visits chiropractor (Ch)}\}$$

We are given $\mathbb{P}((A \cup B)') = 0.08$, $\mathbb{P}(A \cap B) = 0.28$ and $P(A) = P(B) + 0.16$ and we are asked to find $\mathbb{P}(A)$. **Method 1** We can use *Venn Diagrams* again. **Method 2** We can use Principle of Inclusion and exclusion.

Example 1.4. Westwood High has 300 students, and it involves three active clubs: Soccer club, Basketball club and Volleyball club. Except 16 students, everybody else is engaged in at least one of the clubs. There are 130 students enrolled in Soccer Club, 100 students enrolled in Basketball club and 144 students enrolled in Volleyball club. There are 30 students who plays both soccer and basketball, 40 students who play both soccer and volleyball and 32 students who play both basketball and volleyball. Find the number of students who play all.

Example 1.5. Find the probability that among 4 friends, there are two of them who is born on the same month. You may assume probabilty of a person born in any month is $\frac{1}{12}$ (E.g. birthdays are uniformly disributed over months. This is most likely not correct.)

1.1.1 Problems

1. ($1.1-3$ **in PSI**) Draw one card at random from a standard deck of cards. The sample space S is the collection of the 52 cards. Assume that the probability set function assigns $1/52$ to each of the 52 outcomes. Let
   - $A = \{x : x \text{ is a jack, queen, or king}\}$,
   - $B = \{x : x \text{ is a 9, 10, or jack and } x \text{ is red}\}$,

- $C = \{x : x \text{ is a club}\}$,
- $D = \{x : x \text{ is a diamond, a heart, or a spade}\}$.

Find

(a) $P(A)$

(b) $P(B)$

(c) $P(A \cap B)$

(d) $P(A \cup B)$

(e) $P(C \cup D)$

(f) $P(C \cap D)$.

2. For two events A and B, we have $\mathbb{P}(A \cap B^c) = 0.2$, $\mathbb{P}(A^C \cap B^C) = 0.16$. What is $\mathbb{P}(B)$?

3. **(**1.1 − 16 **in PSI)** Let $p_n, \quad n = 0, 1, 2, \ldots$ be the probability that an automobile policyholder will file for n claims in a five-year period. The actuary involved makes the assumption that $p_{n+1} = (1/4)p_n$. What is the probability that the holder will file two or more claims during this period?

1.2 Ch 1.2: Methods of Enumeration

Theorem 1.2 (**Permutation of Different Objects**). *Assume we have n different objects: $O_1, O_2, \ldots, O_n$.*

1. *There are exactly $n! := 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ different ways to* ***permute*** *all of n different objects*
2. *Now assume we want to permute only $r \leq n$ of the objects. Then there exactly $n \cdot (n-1) \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$ ways to* ***permute*** *them. This is usually denoted by ${}_nP_r = \frac{n!}{(n-r)!}$*

Example 1.6. 10 horses are racing in a derby. The first, the second, and the third will be determined and awarded. How many possible outcomes are possible?

Solution

Theorem 1.3 (**Combination of Different Objects**). *Assume we have n different objects: $O_1, O_2, \ldots, O_n$. Now assume we want to* ***choose*** *only $r \leq n$ of the objects. Then there exactly*

$$\frac{n \cdot (n-1) \cdot (n-1) \cdots (n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

ways to choose them. This is usually denoted by $\binom{n}{r} = \frac{n!}{(n-r)!r!}$

Example 1.7. We hava 10 different flavored candies and we want to choose 3 of them. In how many ways we can do it?

Solution

Remark 1.1. Permutation of Letters In general, if we have n objects in total, among which n_i of them are the same for $i = 1, 2, \ldots, k$, there are

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

total ways to permute them. Here of course $n_1 + n_2 + \cdots + n_k = n$.

1.2.1 Problems

1. **(PSI 1.2.5)** How many four-letter code words are possible using the letters in IOWA if
   (a) The letters may not be repeated?
   (A)12 (B)24 (C)48 (D)256 (E)*None*
   (b) The letters may be repeated?
   (A)12 (B)24 (C)48 (D)256 (E)*None*
2. **(PSI 1.2.5-modified)** How many code words up to 5 letters are possible using the letters in IOWA if
   (a) The letters may not be repeated?
   (A)24 (B)48 (C)64 (D)256 (E)*None*
   (b) The letters may be repeated?
   (A)1024 (B)1228 (C)1364 (D)2048 (E)*None*
3. How many 6-digit number can be constructed using $1, 2, 2, 3, 3, 3$?

4. How many 8 letter words we can create (meaningful or not) using the letters of

$$"LAGALAXY"?$$

(A)$8!$ (B)$\frac{8!}{2}$ (C)$\frac{8!}{4}$ (D)$\frac{8!}{6}$ (E)$\frac{8!}{12}$

5. A *round-robin* tournament is being held with n tennis players; this means that every player will play against every other player exactly once.
   (a) How many games are played in total?
   (b) How many possible outcomes are there for the tournament (the outcome lists out who won and who lost for each game)?

6. A *knock-out* tournament is being held with 2^n tennis players. This means that for each round, the winners move on to the next round and the losers are eliminated, until only one person remains. For example, if initially there are $2^4 = 16$ players, then there are 8 games in the first round, then the 8 winners move on to round 2, then the 4 winners move on to round 3, then the 2 winners move on to round 4, the winner of which is declared the winner of the tournament. (There are various systems for determining who plays whom within a round, but these do not matter for this problem.)
   (a) How many rounds are there?
   (b) Count how many games in total are played, by adding up the numbers of games played in each round.
   (c) Count how many games in total are played, this time by directly thinking about it without doing almost any calculation.

   Hint: How many players need to be eliminated?

7. Three people get into an empty elevator at the first floor of a building that has 10 floors. Each presses the button for their desired floor (unless one of the others has already pressed that button). Assume that they are equally likely to want to go to floors 2 through 10 (independently of each other). What is the probability that the buttons for 3 consecutive floors are pressed?

8. **(Old Exam Problem)** At a political meeting there are seven liberals and six conservatives. We choose five people uniformly at random to form a committee. Let A be the event that we end up with a committee consisting of more conservatives than liberals. Find $P(A)$. (You do not need to simplify)

9. **(PSI 1.2.7)** In a state lottery, four digits are drawn at random one at a time with replacement from 0 to 9. Suppose that you win if any permutation of your selected integers is drawn. Give the probability of winning if you select
   (a) 6, 7, 8, 9.

   (A)$\frac{24}{10^4}$ (B)$\frac{24}{10\cdot9\cdot8\cdot7}$ (C)$\frac{4}{10\cdot9\cdot8\cdot7}$ (D)$\frac{12}{10^4}$ (E)$\frac{4^4}{10^4}$

   (b) 6, 7, 8, 8.

   (A)$\frac{24}{10^4}$ (B)$\frac{24}{10\cdot9\cdot8\cdot7}$ (C)$\frac{4}{10\cdot9\cdot8\cdot7}$ (D)$\frac{12}{10^4}$ (E)$\frac{4^4}{10^4}$

   (c) 7, 7, 8, 8.

   (A)$\frac{6}{10^4}$ (B)$\frac{6}{10\cdot9\cdot8\cdot7}$ (C)$\frac{4}{10\cdot9\cdot8\cdot7}$ (D)$\frac{12}{10^4}$ (E)$\frac{24}{10^4}$

   (d) 7, 8, 8, 8.

   (A)$\frac{4}{10^4}$ (B)$\frac{24}{10\cdot9\cdot8\cdot7}$ (C)$\frac{4}{10\cdot9\cdot8\cdot7}$ (D)$\frac{12}{10^4}$ (E)$\frac{4^4}{10^4}$

10. **(PSI 1.2.6)** Suppose that Novak Djokovic (D) and Roger Federer (F) are playing a tennis match in which the first player to win three sets wins the match. Using D and F for the winning player of a set, in how many ways could this tennis match end?

11. **Challenge Problem 1**

    (a) How many triples $a, b, c \in \{1, 2, 3, \ldots, 10\}$ exist so that $a < b < c$?

    (b) How many triples $a, b, c \in \{1, 2, 3, \ldots, 10\}$ exist so that $a \leq b \leq c$?

12. **Challenge Problem 2 (HMMT 2005 General 2)** In how many ways can 8 people be arranged in a line if Alice and Bob must be next to each other, and Carol must be somewhere behind Dan?

1.3 ch 1.3: Conditional Probability

Consider the first example from the first discussion, that we have a class of 60 as shown in the table

Class		
	Boys	Girls
Blue	10	15
Green	25	10

The teacher chooses one student uniform at random in the class but does not tell the students. She wants the students to guess the eye color of S, so that those who guesses correctly will get 3 points extra in the next midterm. At first, betting that S has green eyes makes more sense as there are more 35 students with green eyes compared to the other 25 students with blue eyes. In probabilistic notation, $\mathbb{P}(A) = \frac{35}{60} > \mathbb{P}(A^c) = \frac{25}{60}$ where $A = \{S \text{ has green eyes}\}$. Then, the teacher gives a hint that S is a girl. This changes everything, because now we have more information and can reduce the set of possibilities. In particular, we only need to consider the set of ladies now.

Now we need look at $\mathbb{P}(A|B)$ where $B = \{S \text{ is a girl}\}$.

Say that student is S. Let's define events $A = \{S \text{ is a boy}\}$, $B = \{S \text{ has green eyes}\}$. The sets A and B can be represented as yellow and red colored areas

	Boys	Girls
Blue	10	15
Green	25	10
The set B		

In this restriction, S is more likely to have blue eyes, with 15 to 10 favor. So we write $\mathbb{P}(S \text{ has blue eyes}|B) = \frac{15}{25}$. Define $C = \{S \text{ has blue eyes}\}$, we can also compute this as

$$\mathbb{P}(C|B) = \frac{\mathbb{P}(C \cap B)}{\mathbb{P}(B)} = \frac{\frac{15}{60}}{\frac{25}{60}} = \frac{15}{25}$$

Note that $\mathbb{P}(C|B)$ and $\mathbb{P}(B|C)$ are completely different things.

Example 1.8. Monica throws two dice in a backgammon game. You know that the sum of two dice is 10. What is the probability that one of the dice is 5?

(A) $\frac{1}{6}$ (B) $\frac{1}{5}$ (C) $\frac{1}{4}$ (D) $\frac{1}{3}$ (E) $\frac{1}{2}$

Solution

Example 1.9. (PSI 1.3.7) An urn contains four colored balls: two orange and two blue. Two balls are selected at random without replacement, and you are told that at least one of them is orange. What is the probability that the other ball is also orange?

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{1}{5}$ (D) $\frac{1}{6}$ (E) $\frac{2}{5}$

Solution

1.4 ch 1.4: Independent Events

Now consider a different class with the same game.

	Boys	Girls
Blue	10	20
Green	20	40

Now compute $P(C)$ and $P(C|B)$. Are they different? Nope, you will find both equal to $\frac{2}{3}$, which means that the information B is not useful for us, i.e. knowing event B happens or not does not say anything about event C. This is exactly what *independence* is. If we have $P(C) = P(C|B)$, we say the events B and C are independent.

Before the next problem, let's recall the following theorem:

Theorem 1.4 (**Principle of Inclusion & Exculusion**). *Let A, B, C, D be any three events. Then*

1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
2. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
3. $P(A \cup B \cup C \cup D) = P(A) + P(B) + P(C) + P(D) - P(A \cap B) - P(A \cap C) - P(A \cap D) - P(B \cap C) - P(B \cap D) - P(C \cap D) + P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) - P(A \cap B \cap C \cap D)$
4. *This is true for any number of sets. Let $A_1, A_2, \ldots, A_n$ be events/sets. Let c_k denot the sum of all probabilities of k-intersections. For example,*

$$a_1 = \sum_{i=1}^{n} P(A_i)$$

$$a_2 = \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$a_3 = \sum_{1 \le i < j < r \le n} P(A_i \cap A_j \cap A_r)$$

so that in general we define

$$a_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$$

The we have

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = a_1 - a_2 + a_3 - a_4 + \cdot + (-1)^{n-1} a_n$$

Example 1.10. (PSI 1.3.9) An urn contains four balls numbered 1 through 4. The balls are selected one at a time without replacement. A match occurs if the ball numbered m is the m^{th} ball selected. Let the event A_i denote a match on the i^{th} draw, for $i = 1, 2, 3, 4$.

1. Find $P(A_i)$ for each i.

   (A)1 (B)$\frac{1}{2}$ (C)$\frac{1}{3}$ (D)$\frac{1}{4}$ (E)$\frac{1}{24}$

2. Find $P(A_i \cap A_j)$ for each $i \neq j$.

   (A)$\frac{1}{4}$ (B)$\frac{1}{6}$ (C)$\frac{1}{12}$ (D)$\frac{1}{24}$ (E)$\frac{1}{120}$

3. Find $P(A_i \cap A_j \cap A_k)$ for each $i \neq j \neq k \neq i$.

   (A)$\frac{1}{4}$ (B)$\frac{1}{6}$ (C)$\frac{1}{12}$ (D)$\frac{1}{24}$ (E)$\frac{1}{120}$

4. Compute $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Note that this represent the probability that there is at least one match.
   HINT: Use principle of inclusion & exclusion

   (A)$\frac{1}{4}$ (B)$\frac{2}{3}$ (C)$\frac{3}{5}$ (D)$\frac{5}{8}$ (E)$\frac{17}{24}$

5. Extend this exercise so that there are n balls in the urn. Show that the probability of at least one match is
$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + \frac{(-1)^n}{n!}$$
6. What is the limit of this probability as n increases without bound?

Solution

1.4.1 Problems

1. Each of three football players will attempt to kick a field goal from the 25-yard line. Let A_j denote the event that the field goal is made by player $j, j = 1, 2, 3$. Assume that A_1, A_2, A_3 are mutually independent and that $P(A_1) = 0.5, P(A_2) = 0.7, P(A_3) = 0.6$.
   (a) Compute the probability that exactly one player is successful.
   (b) Compute the probability that exactly two players make a field goal (i.e., one misses).
2. **(Old MT Problem)** Suppose that $P(A) = 0.4$, $P(B) = 0.3$ and $P((A \cup B)') = 0.42$. Are A and B independent?
3. **(Old HW Question)** Show via example that if A, B are independent. Then it is not necessarily true that
$$P[C|A \cap B] = P[C|A]P[C|B]$$
Similarly show that it is not necessarily true that
$$P[A \cup B|C] = P[A|C]P[B|C]$$
4. **(Old MT Problem)** Each of the 6 students is given a fair 6-sided dice. In addition, each student is numbered from 1 to 6. (Write your answers of the following questions using fractions and binomial numbers)
   (a) If the students roll their dice, what is the probability that there is at least one "match" (e.g. student 4 rolls a 4)?
   (b) If you are the first student, what is the probability that at least one of the other five students rolls the same number as you?
   (c) Solve part (b) with the assumption now that the dice is generally not fair so that the probability of rolling i is $p_i > 0$ with $\Sigma_{i=1}^{6} pi = 1$.
5. **(Old MT Problem)** Let A be an event that is independent with itself. What value(s) can $P(A)$, the probability that A happens, might take?

1.5 ch 1.5: Bayes' Theorem

1. **(PSI 1.5.9 - modified)** There is a new diagnostic test for *coronavirus* that occurs in about 10% of the population. The test is not perfect, but will detect a person with the disease 95% of the time. It will, however, say that a person without the disease has the disease about 20% of the time. A person is selected at random from the population, and the test indicates that this person has the disease. What is the conditional probabilities that the person actually has the disease?

   (A) $\frac{1}{3}$ (B) $\frac{9}{100}$ (C) $\frac{19}{55}$ (D) $\frac{15}{66}$ (E) $\frac{33}{95}$

2. Suppose you are playing cards, and you are waiting for your last card to be dealt. Suppose the last card could be an Ace of Spades, a King of Hearts, or two of Clubs. Suppose that you win with a probability of two-thirds if you get the Ace, a probability of one-half if you get the King, and a probability of one-tenth if you get the two of Clubs. Suppose you get each card with equal likelihood.

   (a) What is the probability that you win?

   (b) Suppose you won, what is the probability you got the Ace?

   (c) Suppose you lost, what is the probability that you didn't get the Ace?

3. In a high school, 70 percent of the students has black hair, and the rest has yellow hair. The probability that a student with black hair has glasses is 0.4, and the probability that a student with yellow hair has glasses is 0.7. A student is randomly picked from the crowd.

   (a) Given that the student wears glasses, what is the probability that she has black hair?

   (A) $\frac{7}{10}$ (B) $\frac{7}{9}$ (C) $\frac{3}{7}$ (D) $\frac{4}{7}$ (E) $None$

   (b) Given that the student doesn't wear glasses, what is the probability that she has black hair?

   (A) $\frac{14}{17}$ (B) $\frac{7}{13}$ (C) $\frac{3}{10}$ (D) $\frac{3}{7}$ (E) $None$

4. **(PSI 1.5.9 - modified V2)** There is a new diagnostic test for *coronavirus* that occurs in about 15% of the population. The test is not perfect, but will detect a person with the disease 95% of the time. It will, however, say that a person without the disease has the disease about 30% of the time. A person is selected at random from the population, and the test indicates that this person has the disease. What is the conditional probability that the person actually has the disease?

5. **(Old Midterm Problem)** A hospital receives 70% of its flu vaccine from Company Good and the remainder from Company Evil. Each shipment contains a large number of vials of vaccine. From Company Good, 10% of the vials are ineffective. From Company Evil, 80% are ineffective. A hospital tests $n = 10$ randomly selected vials from one shipment for their effectiveness.

   (a) Compute the probability that exactly two of these 10 vials are ineffective if this shipment comes from Company Good. (You do not need to simplify)

   $P(I \mid G) = \binom{10}{2}\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^8 = \frac{(45)\,(9^8)}{(10)^{10}}$.

   (b) Compute the probability that exactly two of these 10 vials are ineffective if this shipment comes from Company Evil. (You do not need to simplify)

   $P(I \mid E) = \binom{10}{2}\left(\frac{8}{10}\right)^2\left(\frac{2}{10}\right)^8 = \frac{(45)\,(2^{14})}{(10)^{10}}$.

   (c) Compute the probability that exactly two of these 10 vials are ineffective. (You do not need to simplify.)

$$\begin{aligned} P(I) =& P(I \mid G)\,P(G) + P(I \mid E)\,P(E) \\ =& \frac{(45)\,(9^8)}{(10)^{10}} \times \frac{7}{10} + \frac{(45)\,(2^{14})}{(10)^{10}} \times \frac{3}{10} = \frac{(45)\,(9^8)\,(7)\,+\,(45)\,(2^{14})\,(3)}{10^{11}} \end{aligned}$$

(d) Compute the conditional probability that this shipment come from Company Good. (Hint: Bayes theorem might be useful here) (You do not need to simplify.)

By Bayes theorem,

$$
\begin{aligned}
P(G \mid I) =& P(G) \times P(I \mid G) \times (P(I))^{-1} \\
=& \left(\frac{7}{10}\right)\left(\frac{(45)\,(9^8)}{(10)^{10}}\right)\left(\frac{(45)\,(9^8)\,(7)\; +\; (45)\,(2^{14})\,(3)}{10^{11}}\right)^{-1} \\
=& \frac{(7)(45)(9^8)}{(45)\,(9^8)\,(7)\; +\; (45)\,(2^{14})\,(3)} \\
=& \frac{(7)(9^8)}{(9^8)\,(7)\; +\; (2^{14})\,(3)}.
\end{aligned}
$$

1.6 General

1. **(Old MT Problem)** Ten men and five women are meeting in a conference room. Four people are chosen at random to form a committee. (Write your answers of the following questions using fractions and binomial numbers).

   (a) What is the probability the committee consists two men and two women.

   (b) Among the 15 is a couple, David and Mary. What is the probability this couple ends up on the committee of four people?

   (c) What is the probability that David ends up on the committee but Mary does not?

2. For two event A and B in the probability space, $P(A \cup B) = 0.6, P(A) = 0.2, P(B) = 0.5$. Determine if two events are independent?

3. In a lotto 5 numbers are selected without replacement among $\{1, 2, ..., 25\}$. What is the probability that 20 and 23 are among those selected? The answer should be a fraction in a form $\frac{1}{k}$ where k is a whole number.

2 Ch2: Discrete Distribution & Random Variables

2.1 Ch 2.1

2.1.1 Problems

1. For each of the following, determine the constant c so that $f(x)$ satisfies the conditions of being a pmf for a random variable X, and then depict each pmf as a line graph:

   (a) $f(x) = x/c$, $x = 1, 2, 3, 4$. (A)1 (B)3 (C)4 (D)6 (E)10

   (b) $f(x) = cx$, $x = 1, 2, 3, ..., 10$. (A)$\frac{1}{3}$ (B)$\frac{1}{10}$ (C)$\frac{1}{45}$ (D)$\frac{1}{55}$ (E)$\frac{1}{100}$

   (c) $f(x) = c(\frac{1}{4})^x$, $x = 1, 2, 3, \ldots$ (A)3 (B)$\frac{1}{3}$ (C)4 (D)$\frac{1}{4}$ (E)1

   (d) $f(x) = \frac{c}{(x)(x+1)}$, $x = 2, 3, 4, \ldots$
   Hint: Write $f(x) = c/x - c/(x+1)$.

2.2 Ch 2.2 - Expectation

2.2.1 Problems

1. **PSI 2.4.1-modified** An urn contains 7 red and 11 white balls. Draw one ball at random from the urn. Let $X = 1$ if a red ball is drawn, and let $X = -1$ if a white ball is drawn. Give the pmf, mean, and variance of X.

2. **PSI-2.2.5** Let the random variable X be the number of days that a certain patient needs to be in the hospital. Suppose X has the pmf

$$f(x) = \frac{5-x}{10}, \quad x = 1, 2, 3, 4$$

   If the patient is to receive \$200 from an insurance company for each of the first two days in the hospital and \$100 for each day after the first two days, what is the expected payment for the hospitalization? (A)260 (B)300 (C)310 (D)350 (E)360

3. On average, how many rolls we need to throw a fair dice to get all 6 outcomes?

   (A)6 (B)36 (C)12 (D)14.7 (E)$\frac{144}{7}$

4. **(Old MT problem)** Dol Amroth is a new colony of the Kingdom of Numenor. As the head of the colony, Lord Imrahil asks King Tar-Calmacil to send them a yearly budget. The chief financial advisor of the King proposes that the Dol Amroth should be send at least 3 (thousand) gold coins every year, but their needs should also be addressed. When asked to report on potential budgetary needs, Lord Imrahil responded that they might need 1, 2, 3, 4, 5 or 6 (thousand) gold coins, with mass probabilities of

$$f(x) = \frac{3 - |x-3|}{10}$$

   for $x = 1, 2, 3, 4, 5$ and $f(6) = 1/10$. So, the king decides to send 3 (thousand) gold coins or in the amount of the needs of the colony, whichever is largest.

   (a) What is the probability that Lord Imrahil will recieve 4 (thousand) or more gold coins?

   (b) What is the expected value and variance of the gold coins Lord Imrahil will receive?

5. **(Old HW Problem)** Suppose I have 5 different colored pairs of socks. I draw one sock at random until I get a matching pair. What is the expected number of socks that I have to draw.

6. A drunken man starts at zero on real line. At each step he picks his direction uniformly random, left or right, and moves length one. Eg, if the is at point x at step k, in his next step he'll move $x+1$ or $x-1$ with $1/2$ probability each. Define the random variable D_m as the drunkard's distance from his point of origin after m steps. Prove that $E[D_m^2] = m$

2.3 Ch 2.3 - Special Expectations

2.3.1 Problems

1. **Random Variable Example** Two dice are thrown: D_1& D_2. Let random variable X be the sum of numbers facing up. Find
   (a) The pmf of X
   (b) $E(X)$
   (c) $M_X(t)$, meaning moment generating function of X.

2. **(Old Quiz Problem)** X is a discrete R.V. with moment generating function
$$M(t) = \frac{3e^t}{4}(1 - \frac{e^t}{4})^{-1}$$
   (a) Find the support and pmf of X (A)1 (B)$\frac{4}{3}$ (C)$\frac{3}{4}$ (D)$\frac{8}{3}$ (E)$\frac{9}{4}$
   (b) Compute the mean of X. (A)1 (B)$\frac{2}{3}$ (C)$\frac{3}{2}$ (D)$\frac{4}{9}$ (E)$\frac{9}{4}$
   (c) Compute $Var(X)$

   HINT: X is actually geometric R.V. with parameter p. Finding p is the most useful. Remark: This problem is related to PSI-2.3.8

3. **(Final 20 Summer)** Balin and Dwalin are two brothers, playing a game with a slightly crooked coin, with $P(H) = 0.6$ and $P(T) = 0.4$. The coin is tossed 10 times (each toss is independent from others) and in any turn it shows heads, it is tossed again. The brothers are counting the cases where the coin is tossed twice and the second toss, too, is heads. Here is an example scenario:
$$T(HT)TTTTTTT(HH)$$
   The count here will be 1, as there are only two cases where the coin showed heads and in only one such case, the second toss was also heads. Thus, only the 10th turn is counted.

   Let X be the counted number of such double heads. What is the support, pmf, expected value and variance of X?

4. **(Old MT problem)** Suppose that a coin is not fair so that the probability of obtaining a head is $p \in (0, 1)$.
   (a) On average, how many flips are needed to obtain a head?
   (b) Find the probability the first time obtaining a head is an even number. Your final answer must not be an infinite series.

2.4 Ch 2.4 - Binomial Distribution

2.4.1 Problems

1. **(Old MT problem)** There are three types of coins A, B, C: coin A tosses head with probability p_1, coin B tosses head with probability p_2, coin C tosses head with probability p_3. The experimenter selects one of the three coins at random (with probability 1/3 each), and then tosses it independently 6 times. Given that the experimenter sees 4 many heads and 2 many tails, what is the probability that he has chosen coin A?

2. **(Old Final Problem)** 6 fair dice are rolled. 6 numbers are facing up. What is the probability that there are exactly 2 of the numbers less or equal than 2?

3. **(Old MT Problem, 20Summer)** Let X be a Binomial random variable with $E[X] = 30$ and $Var(X) = 21$. Calculate $F(4) - F(3)$, where F is the CDF of X.

2.5 Ch 2.5 - Hypergeometric Distribution

2.5.1 Problems

1. **(Old MT Problem)** A group of 10 men and 10 women is randomly divided into 2 groups of size 10 each. What is the probability that both groups will have the same number of men?

2. **(Old Final Problem)** For a final exam, your professor gives you a list of 10 items to study. He indicates that he will choose 5 for the actual exam. You will be required to answer 3 of those correctly to pass. You decide to study 6 of the 10 items. What is the probability that you will pass the exam?

2.6 Ch 2.6 - Negative Binomial Distribution

2.6.1 Problems

1. **(Old Quiz Problem-PSI 2.6.10-modified**) Red Rose Tea randomly began placing one of 25 English porcelain miniature figurine in a 100-bag box of tea, selecting from ten figurine in the American Heritage series. A customer, a big fan of George Washington, wants to have three copies of George Washington figurine; one for his collection, one to give to an acquaintance, and one for his own entertainment. On the average, how many boxes of tea must be purchased by a customer to obtain three copies of George Washington figurine?

2. You throw a dice repeatedly. Let X be the number of throws until you get three $6s$. What is the expected number throws?

3. You have an unfair coin with $P(Heads) = 1/3$, and you keep tossing until you see 3 heads (not necessarily back to back).

   (a) Find the probability that the third head is observed on the fifth toss.

   (b) On average, how many tosses needed to see three heads?

2.7 Ch 2.7 - The Poisson Distribution

2.7.1 Problems

1. **(PSI 2.7.5)** Flaws in a certain type of drapery material appear on the average of one in 150 square feet. If we assume a Poisson distribution, approximate the probability of at most one flaw appearing in 225 square feet.

   (A)0.06 (B)0.12 (C)0.33 (D)0.56 (E)0.56

2. **(Old Final Problem)** Each of 1000 people picks a positive integer number between 1 and 1000 (including 1 and 1000), uniformly at random and independently of each other. Approximate the probability that exactly 3 of them picks a number less or equal than 5.

3. **(Old MT problem)** Let us consider an unfair coin which tosses head and tail with probability 0.01 and 0.99, respectively.

   (a) Toss this coin independently, and let X be the first time that head appears. Compute the mean and variance of X.

   (b) Toss this coin for 500 times independently. Use a Poisson limit theorem to compute (approximately) the probability that exactly 5 many head appears.

4. Suppose customers arrive at a store according to a Poisson process with an average of 10 customers per hour.

   (a) What is the probability no customers show up in the first ten minutes?

   (b) Let random variable X = number of minutes until 2 customers show up at the store. What is $E[X]$?

5. **(Old MT problem)** Roll a pair of dice 100 times (so on each of the 100 rolls you roll two dice). Approximate the probability you never get "snake eyes" (snake eyes means you get a 1 on both dice).

6. **(Old HW problem)** Raindrops are falling at an average rate of 10 drops per square inch per minute. What would be a reasonable distribution to use for the number of raindrops hitting a particular region measuring 4 square inches in a minute? Why? Using your chosen distribution, compute the probability that the region has no rain drops in a given 10-second time interval.

7. **(Old MT problem)** Raindrops are falling at an average rate of 30 drops per square inch per minute.

   (a) What would be a reasonable distribution to use for the number of raindrops hitting a particular region measuring 5 square inches in a minute? (write the name of dist/corresponding parameters).

   (b) What would be a reasonable distribution for the waiting time of the first raindrop hitting the region measuring 2 square inches (write the name of dist/corresponding parameters)?

8. Assume that a policyholder is four times more likely to file exactly two claims as to file exactly three claims. Assuming that the number X of claims of this policyholder is Poisson, determine $Var(X)$.

2.8 Ch2-Misc

2.8.1 Problems

1. **(Quiz - 20 Summer)** Let Z be the random variable with support $S_Z = \{1, 2, 3\}$ and pmf $f(z) = z/6$, for $z \in S_Z$.

   (a) Find the MGF (moment generating function) of Z, $M_Z(t)$.

   (b) Using $M_Z(t)$, calculate $E[Z^3]$.

3 Ch3: Continuous Distributions

3.1 Ch 3.1: RVs of Cts Typ

3.1.1 Problems

1. **(Old Final Problem)** A continuous random variable X has probability density function $f(x) = 3x^2$ for $0 \leq x \leq 1$, and $f(x) = 0$ for everywhere else. Compute $P(X > \frac{1}{2}|X > \frac{1}{3})$
2. **(PSI 3.1.15)** The life X (in years) of a voltage regulator of a car has the pdf

$$f(x) = \frac{3x^2}{7^3} e^{-(x/7)^3}$$

for $0 < x < \infty$.

   (a) What is the probability that this regulator will last at least 7 years?

$$\text{(A)}e^{-1} \qquad \text{(B)}e^{-7} \qquad \text{(C)}e^{-1/7} \qquad \text{(D)}\frac{1}{7} \qquad \text{(E)}\frac{316}{343}$$

   (b) Given that it has lasted at least 7 years, what is the conditional probability that it will last at least another 3.5 years?

$$\text{(A)}e^{-1} \qquad \text{(B)}e^{-19/8} \qquad \text{(C)}e^{-27/8} \qquad \text{(D)}\frac{7}{8} \qquad \text{(E)}\frac{27}{343}$$

3. **(Old Quiz Problem)** Let X be a continuous random variable with uniform distribution $U([3, 4])$.

   (a) Compute $\mu = EX$

$$\text{(A)}3 \qquad \text{(B)}3.5 \qquad \text{(C)}4 \qquad \text{(D)}7 \qquad \text{(E)}49$$

   (b) Compute $M(t)$

$$\text{(A)}t^7 \qquad \text{(B)}e^{7t/2} \qquad \text{(C)}\frac{e^{7t/2}}{12} \qquad \text{(D)}\frac{e^{4t} - e^{3t}}{12} \qquad \text{(E)}\frac{e^{4t} - e^{3t}}{t}$$

   (c) Compute $\sigma^2 = Var(X)$

$$\text{(A)}1 \qquad \text{(B)}12 \qquad \text{(C)}7 \qquad \text{(D)}\frac{1}{7} \qquad \text{(E)}\frac{1}{12}$$

4. **(Old Final Problem)** A continuous random variable X has probability density function $f(x) = c(3x^2 + 2x)$ for $0 \leq x \leq 2$, and $f(x) = 0$ for everywhere else (here c is a constant value). Compute $P(X < \frac{3}{2}|X > \frac{1}{2})$. The answer should not depend on c.
5. **(Old MT Problem)** Let the random variable X have $E(X^2) = 2$ and $E(\frac{1}{X^2}) = 1$.

   (a) Compute $E(\frac{(X^2+1)^2}{X^2})$

   (b) Find a real number b that minimizes $E(X - \frac{b}{X})^2$

6. **(Old MT Problem, 20S)** Suppose a RV X has the cdf

$$F_X(x) = \frac{x}{x + 1}$$

for $0 < x < \infty$ and $F(x) = 0$ everywhere else.

   (a) Find pdf of X, $f_X(x)$

   (b) Compute $E[(X + 1)^2 e^{-2X}]$

3.2 Ch 3.2 Various Distributions

3.2.1 Problems

1. **(PSI 3.2.24)** Let the random variable X be equal to the number of days that it takes a high-risk driver to have an accident. Assume that X has an exponential distribution. If $\mathbb{P}(X < 50) = 0.25$, compute $\mathbb{P}(X > 100|X > 50)$.

$$\text{(A)}\frac{1}{4} \qquad \text{(B)}\frac{1}{2} \qquad \text{(C)}\frac{\sqrt{2}}{2} \qquad \text{(D)}\frac{3}{4} \qquad \text{(E)}1 - \frac{\sqrt{2}}{2}$$

3.3 Ch 3.3: Normal Distribution

3.3.1 Problems

1. **(PSI 3.3.1-modified)** If Z is $N(0,1)$, find the following probabilities using standard normal table:
   (a) $\mathbb{P}(Z \le 1.2)$
   (b) $\mathbb{P}(Z \le 0.87)$
   (c) $\mathbb{P}(0.53 < Z \le 2.06)$
   (d) $\mathbb{P}(Z \le -1)$
   (e) $\mathbb{P}(-0.79 \le Z < 1.52)$
   (f) $\mathbb{P}(Z > -1.77)$
   (g) $\mathbb{P}(Z > 2.89)$
   (h) $\mathbb{P}(|Z| < 1.96)$
   (i) $\mathbb{P}(|Z| < 2)$ (A)0.453 (B)0.671 (C)0.881 (D)0.954 (E)0.98
2. **(PSI 3.3.7)** If X is $\mathcal{N}(3,4)$, find $\mathbb{P}(|X| < 4)$

   (A)0.13 (B)0.25 (C)0.36 (D)0.56 (E)0.69

3. **(PSI 3.3.7)** If X is $\mathcal{N}(650,400)$, find
   (a) $\mathbb{P}(600 \le X < 660) \approx$ (A)0.68 (B)0.72 (C)0.83 (D)0.94 (E)$None$
   (b) A constant $c > 0$ such that $\mathbb{P}(|X - 650| \le c) = 0.9544$
   (A)12.1 (B)24.7 (C)36 (D)39.2 (E)44.9

   Solution Write $X = \mathcal{N}(650,400) = 650 + 20Z$. Note that here $\mu = 650, \sigma = 20$, and Z is the standard normal $Z = \mathcal{N}(0,1)$. Here, note the distinction with the standard deviation $\sigma = 20$ and variance $\sigma^2 = 400$
4. **(3.3-6 in PSI)** If the mgf of RV X is $M(t) = e^{166t+200t^2}$ for $-\infty < t < \infty$, find
   i $\mathbb{E}X$
   ii $Var(X)$
   iii $\mathbb{P}(160 < X < 200)$

4 Ch4: Bivariate/Joint Distributions

4.1 Ch 4.1: Bivariate/Joint Distributions of Discrete Type

4.1.1 Problems

1. **(PSI example 4.1.3)** The joint pmf of $X\&Y$ is

$$f(x,y) = c(x+y) \text{ for } x = 1,2,\ y = 1,2,3$$

   (a) Find c. (A)6 (B)$\frac{1}{6}$ (C)12 (D)21 (E)$\frac{1}{21}$

   (b) Find marginal pmf $f_X(x)$. (A)$\frac{2x+y}{9}$ (B)$\frac{2x+1}{7}$ (C)$\frac{x+1}{7}$ (D)$\frac{x+2}{7}$ (E)$\frac{x+3}{9}$

   (c) Find marginal pmf $f_Y(y)$. (A)$\frac{2y+3}{21}$ (B)$\frac{y+6}{21}$ (C)$\frac{y+1}{7}$ (D)$\frac{2y+1}{14}$ (E)$\frac{3y+1}{21}$

   (d) Find $P(X > Y)$ (A)$\frac{2}{21}$ (B)$\frac{1}{7}$ (C)$\frac{4}{21}$ (D)$\frac{5}{21}$ (E)$\frac{2}{7}$

   (e) Find covariance of $X\&Y$. Are they indepenedent?

   (f) Find correlation coefficient of $X\&Y$, i.e. ρ.

   (g) Find the conditional probability distribution $g(x|y)$.

4.2 Ch 4.2: Correlation Coefficient

4.3 Ch 4.3: Conditional Distributions

4.3.1 Problems

1. **(PSI 4.3-10**) Let $f_X(x) = 1/10$, for $x = 0, 1, 2, ..., 9$, and $h(y|x) = 1/(10-x)$, for $y = x, x+1, ..., 9$. Find
   (a) $f(x, y)$
   (b) $f_Y(y)$.
   (c) $E(Y|x)$.
2. Let $X \sim Unif(2, 4)$[1] and let $Y \sim Pois(X)$[2]. Using *Laws of Total Expectation and Variance*, compute $\mathbb{E}Y$ and $Var(Y)$.

   Hint: Recall *Laws of Total Expectation and Variance*:
   $$\mathbb{E}Y = \mathbb{E}(\mathbb{E}(Y \mid X))$$
   $$Var(Y) = \mathbb{E}(Var(Y \mid X)) + Var(\mathbb{E}(\mathbb{Y} \mid \mathbb{X}))$$
3. Roll a fair dice, say the random variable D is the outcome. Then throw a fair coin D times. Say X is the number of heads you see. Compute $P(X = 5)$. Find EX and $Var(X)$.
4. **(PSI 4.3-5**) Let X and Y have a trinomial distribution with $n = 2$, $p_X = 1/4$, and $p_Y = 1/2$. Find $E(Y|x)$.

[1] Meaning X has distribution uniform distribution $Unif(2, 4)$
[2] More precisely, this means $Y \mid X = x$ (Y given $X = x$) has distribution $Pois(x)$

4.4 Ch 4.4: Bivariate/Joint Distributions of Cts Type

4.4.1 Problems

1. **(PSI 4.4.1-modified)** Let $f(x, y) = cx^2y^3$, $-1 \leq x \leq 1, 0 \leq y \leq 1$, be the joint pdf of X and Y.
   (a) Compute c.
   (b) Find $f_X(x)$, the marginal probability density function.
   (c) Find $f_Y(y)$.
   (d) Are the two random variables independent? Why or why not?
   (e) Compute the means and variances of X and Y.
   (f) Find $\mathbb{P}(X \leq Y)$.
2. **(Old MT Problem)** Let (X, Y) have the joint pdf $f(x, y) = 3(2 - x)y, 0 < y < 1, y < x < 2 - y$.
   (a) Sketch the region for which $f(x, y) > 0$.
   (b) Find marginal f_X (be sure to include its support).
   (c) Find $Cov(X, Y)$.
   (d) Find $P(X + Y \leq 1)$.

4.5 Ch 4.5: The Bivariate Normal Distributions

4.5.1 Problems

1. **(Old Midterm Problem)** Assume that the high school grade point average of a student, X, and the college freshman gpa of a student, Y, are distributed according to a bivariate normal distribution with parameters $\mu_X = 3$, $\mu_Y = 4.3$, $\sigma_X = 0.4$, $\sigma_Y = 1.25$, and $E[XY] = 13.3$

   (a) Find the correlation coefficient ρ of X and Y.
   (A)0.2 (B)0.25 (C)0.4 (D)0.6 (E)0.8

   (b) Are X and Y independent random variables? Why or why not?

   (c) Compute the probability $P(3.4 < X < 4.2)$

   (A)0.1574 (B)0.2215 (C)0.324 (D)0.4512 (E)0.52

   (d) Compute the probability $P(7.8 < Y < 8.55 \mid X = 5)$
   HINT: Recall $\mu_{Y|x} = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)$ **and** $\sigma_{Y|x} = \sigma_Y\sqrt{1-\rho^2}$

5 Ch5: Distributions of Functions of RVs

5.1 Functions of One RV

Change of Variable Technique - Why does it work? Recall the theorem. X is random variable, u is stricly[3] monotone function so that $v = u^{-1}$ exist. Another random variable defined as $Y = v(X)$. Then $f_Y(y) = f_X(v(y))|v'(y)|$.

5.1.1 Problems

1. **(5.1.3 in PSI)** Let X have the density $f(x) = 4x^3,\ 0 < x < 1$. Find the density of $Y = x^2$.
(A)1 (B)$2y$ (C)$3y^2$ (D)$4y^3$ (E)$5y^4$

2. **(5.1.7 in PSI)** Assume $\theta \in \mathbb{R}^+$. The density of X is $f_X(x) = \theta x^{\theta-1}$ for $0 < x < 1$. Let $Y = -2\theta \ln X$. How Y is distributed?

3. **(Old MT Problem)** Given a constant c, suppose X is a continuous RV with pdf

$$f(x) = c(x+3), \qquad -2 < x < 1$$

   (a) Find c.

   (b) Let $Y = |X+1|$. Find the support and pdf of Y.

4. (2020 Summer Quiz5 P1) Let X be exponential random variable with $\lambda = 1$.

   (a) (4 points) Define $Y = \sqrt{X}$. Specify the support of Y and compute its density.

   Sol: The support of X is $(0, \infty)$, so the support of $Y = \sqrt{X}$ is also $(0, \infty)$.

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\sqrt{X} \le y) = \mathbb{P}(X \le y^2) = F_X(y^2)$$

   Recall that $X \sim exp(1)$, so $F_X(x) = 1 - e^{-x}$, thus $F_Y(y) = F_X(y^2) = 1 - e^{-y^2}$. Then

$$f_Y(y) = \frac{\partial}{\partial y} F_Y(y) = \frac{\partial}{\partial y}(1 - e^{-y^2}) = \boxed{2ye^{-y^2}}$$

   Sol 2: We can also use *Change of Variable Technique* in this problem. $u(x) = \sqrt{x}$ is monotone increasing on $(0, \infty)$, so we can apply it. $v(y) = u^{-1}(y) = y^2$, thus $f_Y(y) = f_X(v(y))|v'(y)| = f_X(y^2)|\frac{\partial}{\partial y} y^2| = \boxed{e^{-y^2} 2y}$

   (b) (6 points) Define $Z = X^2 + 2X$. Specify the support of Z and compute its density.

   Sol: The support of X is $(0, \infty)$, so the support of $Z = X^2 + 2X$ is also $(0, \infty)$.

$$\begin{aligned} F_Z(z) &= \mathbb{P}(Z \le z) \\ &= \mathbb{P}(X^2 + 2X \le z) \\ &= \mathbb{P}((X+1)^2 \le z+1) \\ &= \mathbb{P}(X + 1 \le \sqrt{z+1}) \\ &= \mathbb{P}(X \le \sqrt{z+1} - 1) \\ &= F_X(\sqrt{z+1} - 1) \end{aligned}$$

   Recall that $X \sim exp(1)$, so $F_X(x) = 1 - e^{-x}$, thus $F_Z(z) = F_X(\sqrt{z+1} - 1) = 1 - e^{1-\sqrt{z+1}}$. Then

$$f_Z(z) = \frac{\partial}{\partial z} F_Z(z) = \frac{\partial}{\partial z}(1 - e^{1-\sqrt{z+1}}) = \boxed{\frac{1}{2\sqrt{z+1}} e^{1-\sqrt{z+1}}}$$

[3] Do we need stricly?

5. **(Old Final Problem)** Company *Acne* provides earthquake insurance. The premium X is modeled by the pdf

$$f(x) = xe^{-x}; \qquad 0 < x < \infty,$$

while the claims Y have the pdf

$$g(y) = e^{-y}; \qquad 0 < y < \infty.$$

If X and Y are independent, find the pdf of $Z = X/Y$.
(Hint: You can use the formula $\int_0^\infty u^2 e^{-u}\,du = 2$ from the lecture.)

(A) $\frac{2z}{(z+1)^3}$ (B) $\frac{3z+1}{(z+1)^4}$ (C) $\frac{1}{z+e}$ (D) $\frac{2}{z+e^2}$ (E) ze^{-z}

Solution We will use the cdf method to solve this problem. First note that the support of Z is the positive reals $[0, \infty)$ since both X and Y are positive reals. Let $H(z)$ be the cdf of Z. Then, for any $z \geq 0$,

$$\begin{aligned}
H(z) =& P(Z \leq z) = P(X/Y \leq z) = P(X \leq zY) \\
=& \int_0^\infty \int_0^{zy} (xe^{-x})(e^{-y})\,dx\,dy \\
=& \int_0^\infty \int_0^{zy} x\,e^{-x-y}\,dx\,dy.
\end{aligned}$$

Since the pdf $h(z)$ of Z is the derivative of $H(z)$, we have

$$\begin{aligned}
h(z) =& \frac{\partial}{\partial z} H(z) = \int_0^\infty \frac{\partial}{\partial z} \left(\int_0^{zy} x\,e^{-x-y}\,dx \right) dy \\
=& \int_0^\infty (y)\,[(zy)\,e^{-(zy)-y}]\,dy \\
=& z \int_0^\infty y^2 e^{-(z+1)y}\,dy.
\end{aligned}$$

Substituting $u = (z+1)y$ and $du = (z+1)dy$, we get

$$\begin{aligned}
h(z) =& z \int_0^\infty \left(\frac{u}{z+1} \right)^2 e^{-u} \left(\frac{du}{z+1} \right) \\
=& \frac{z}{(z+1)^3} \int_0^\infty u^2\,e^{-u}\,du \\
=& \frac{z}{(z+1)^3}\,(2) = \frac{2z}{(z+1)^3}.
\end{aligned}$$

5.2 Transformations of Two RVs

5.2.1 Problems

1. **(5.2.2 in PSI)** Let X_1, X_2 denote two independent random variables, each with $\chi^2(2)$ distribution.

   i Find the joint pdf of $Y_1 = X_1$ and $Y_2 = X_1 + X_2$. Note that the support of (Y_1, Y_2) is $\{(y_1, y_2) \mid 0 < y_1 < y_2 < \infty\}$.

   ii Find the marginal pdf of each Y_1 and Y_2.

   iii Are Y_1 and Y_2 independent?

2. **(5.2.2 in PSI - modified)** Let X_1, X_2 denote two independent random variables, each with $\chi^2(2)$ distribution. Find the joint pdf of $Y_1 = X_1X_2$ and $Y_2 = X_1^2X_2$. Note that the support of (Y_1, Y_2) is $\{(y_1, y_2) \mid 0 < y_1, y_2 < \infty\}$.

5.3 Several RVs

5.3.1 Problems

1. **(Old Final Problem)**Let X and Y be independent exponential random variables, both with mean 1. Let $Z = Y - X$.

   (a) Let $f_Z(z)$ be the pdf of Z. Use the cdf method to calculate $f_Z(z)$ for $z > 0$. (Of course, $f_Z(z)$ is also defined for $z \leq 0$, but for the sake of this problem you are not required to compute those densities.)

   (b) Find the mgf of Z.
   Hint. Find the mgf of Y and $-X$ and then use that $Z = Y + (-X)$. Be sure to specify where the mgf is defined.

5.4 MGF Technique

5.4.1 Problems

1.

5.5 Random Functions Associated with Normal Distribution

5.5.1 Problems

1. **(2020 Summer Quiz5)** Company A produces pencils of length 8 inches on average. The machines in the production are old, so the lengths of the pencils are distributed according to normal distribution $\mathcal{N}(8, 1.44)$[4]. Let X_i denote the length of i^{th} pencil produced. You may assumme $\{X_i\}_{i=1}^{\infty}$ are independent (i.e, the length of each pencil are independent). Define

$$\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$$

as the average length of first n pencils produced.

(a) (5 points) Find m if $Var(\bar{X}_m) = 0.005$

Sol: If $\{X_i\}$ are iid normal variables with mean μ and variance σ^2, then

$$\bar{X}_m = \frac{X_1 + X_2 + \cdots + X_m}{m} \sim \mathcal{N}(\mu, \frac{\sigma^2}{m})$$

I.e., $Var(\bar{X}_m) = \frac{Var(X_1)}{m} = \frac{1.44}{m} = 0.005$. This is satisfied when $m = \frac{1.44}{0.005} = \boxed{288}$.

(b) (5 points) Compute $\mathbb{P}(\bar{X}_{25} < 7.76)$.

Sol $\bar{X}_{25} \sim \mathcal{N}(8, \frac{1.44}{25}) = \mathcal{N}(8, (0.24)^2)$. So wen can write $\bar{X}_{25} = 8 + 0.24Z$ where $Z \sim \mathcal{N}(0, 1)$. Then

$$\mathbb{P}(\bar{X}_{25} < 7.76) = \mathbb{P}(8 + 0.24Z < 7.76) = \mathbb{P}(Z < \frac{7.76 - 8}{0.24}) = \mathbb{P}(Z < -1) = \Phi(-1) \approx \boxed{0.1586}$$

2. **(5.5.3 in PSI)** Let X equal to the widest diameter (in milimeters) of the fetal head measured between the 16th and 25th weaks of pregnancy. Assume that the distribution of X is $\mathcal{N}(46.28, 40.96)$. Let $\bar{X}$ be the sample mean of a random sample of $n = 16$ observations of X.

(a) Give values of $\mathbb{E}\bar{X}$ and $Var(\bar{X})$. How $\bar{X}$ is distributed?

(b) Find $\mathbb{P}(44.42 \le \bar{X} \le 48.98)$

[4] This means that the length can also be negative. But since $\mathbb{P}(\mathcal{N}(8, 1.44) < 0) \approx 1.3083924686052994 * 10^{-11}$ is very close to zero you may disregard this case

5.6 CLT

5.6.1 Problems

1. **(Old Final Problem)** Let X be the number of hours a randomly selected child plays PS4 in a month, and let Y be the number of hours the same child plays Nintendo Switch in a month. From personal experience, it is known that

$$\mu_X = 30,\ \mu_Y = 50,\ \sigma_X^2 = 52,\ \sigma_X^2 = 64,\ Cov(X, Y) = 14$$

   (a) Compute the mean and variance of $X + Y$

   (b) Twenty-five children are selected at random. Let Z equal to the total number of hours these 25 children play PS4 or Nintendo Switch in the next month. Use central limit theorem to approximate $P(1940 < Z < 2180)$. (Hint: Use the cdf of standard normal.)