Evans PDE Solutions for Ch2 and Ch3

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SOLUTIONS OF CHAPTER 2

1. Consider the function $z : \Re \to \Re$ for fixed $x \in \Re^n$ and $t \in (0, \infty)$

$$z(s) = u(x + bs, t + s)e^{cs}$$

Then

$$\dot{z}(s) := \frac{\partial z}{\partial s} = e^{cs}(b \cdot D_x u(x + sb, t + s) + u_t(x + sb, t + s) + cu(x + sb, t + s)) = 0$$

by the condition given by the problem. Therefore, z is a constant function with respect to s. Finally, by using the fact that $u = g$ on $\Re^n \times \{t = 0\}$, we conclude that

$$u(x, t) = z(0) = z(-t) = u(x - tb, 0)e^{-ct} = g(x - tb)e^{-ct}$$

2. Let $O = (a_{ij})$ and $\phi(x) = O \cdot x$. Then it's clear that $D\phi(x) = O$. Since v is defined to be $v(x) = u(O \cdot x) = (u \circ \phi)(x)$, we calculate

$$Dv(x) = Du(\phi(x)) \cdot D\phi(x) = Du(O \cdot x) \cdot O$$

Then $v_{x_i}(x) = \sum_{j=1}^{n} u_{x_j}(\phi(x))a_{ji}$. Note that by the same calculations, we have

$$D(u_{x_j} \circ \phi)(x) = Du_{x_j}(\phi(x)) \cdot O$$

Therefore

$$v_{x_i x_i}(x) = \frac{\partial}{\partial x_i} v_{x_i}(x) = \sum_{j=1}^{n} \frac{\partial}{\partial x_i} u_{x_j}(\phi(x))a_{ji} = \sum_{j=1}^{n} a_{ji}(\sum_{k=1}^{n} u_{x_j x_k}(\phi(x))a_{ki}) = \sum_{j,k=1}^{n} a_{ji}a_{ki}u_{x_j x_k}(\phi(x))$$

The laplacian of v is

$$\Delta v(x) = \sum_{i=1}^{n} v_{x_i x_i}(x) = \sum_{i=1}^{n}(\sum_{j,k=1}^{n} a_{ji}a_{ki}u_{x_j x_k}(\phi(x))) = \sum_{j,k=1}^{n}(u_{x_j x_k}(\phi(x))\sum_{i=1}^{n} a_{ji}a_{ki})$$

Since O is an orthogonal matrix,

$$\sum_{i=1}^{n} a_{ji}a_{ki} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{otherwise} \end{cases}$$

Lastly, we conclude our proof with

$$\Delta v(x) = \sum_{j=1}^{n}(u_{x_j x_j}(\phi(x)) = \Delta u(\phi(x)) = 0$$

since u is harmonic.

3. We'll modify the mean value property for harmonic functions. It is assumed that the reader know the proof of the mean value property(Check Evans PDE 2.2.2, Mean Value Formulas). As in the proof in the book, define

$$\phi(s) = \frac{1}{n\alpha(n)s^{n-1}} \int_{\partial B(0,s)} u(y)dS(y)$$

Then the book proves

$$\phi'(s) = \frac{1}{n\alpha(n)s^{n-1}} \int_{B(0,s)} \Delta u(y)dy$$

Since $\Delta u = -f$ in $B(0,r)$

$$\phi'(s) = \frac{-1}{n\alpha(n)s^{n-1}} \int_{B(0,s)} f(y)dy$$

By Fundamental Theorem of Calculus,

$$\phi(r) - \phi(\epsilon) = \int_\epsilon^r \phi'(s)ds$$

for any $\epsilon > 0$. Note that

$$\begin{aligned}
\lim_{\epsilon\to 0} \phi(\epsilon) &= \lim_{\epsilon\to 0} \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(0,\epsilon)} u(y)dS(y) \\
&= \frac{1}{m(\partial B(0,\epsilon))} \int_{\partial B(0,\epsilon)} u(y)dS(y) \\
&= u(0) \qquad (3.1)
\end{aligned}$$

Let's calculate $\int_\epsilon^r \phi'(s)ds$

$$\begin{aligned}
\int_\epsilon^r \phi'(s)ds &= \int_\epsilon^r \frac{-1}{n\alpha(n)s^{n-1}} \int_{B(0,s)} f(y)dyds \\
&= \frac{-1}{n\alpha(n)} \int_{B(0,r)} f(y) \int_{max(|y|,\epsilon)}^r \frac{1}{s^{n-1}}dsdy
\end{aligned}$$

In the last equality, we changed the order of integration. Note that

$$\int_\zeta^r \frac{1}{s^{n-1}}ds = \frac{-1}{(n-2)s^{n-2}}\Bigg|_\zeta^{s=r} = \frac{1}{n-2}(\frac{1}{\zeta^{n-2}} - \frac{1}{r^{n-2}}) \qquad (3.2)$$

Therefore

$$\begin{aligned}
\int_\epsilon^r \phi'(s)ds &= \frac{1}{n\alpha(n)} \int_{B(0,r)} f(y) \int_{max(|y|,\epsilon)}^r \frac{1}{s^{n-1}}dsdy \\
&= \frac{-1}{n\alpha(n)}(\int_{B(0,r)-B(0,\epsilon)} f(y) \int_{|y|}^r \frac{1}{s^{n-1}}dsdy + \int_{B(0,\epsilon)} f(y) \int_\epsilon^r \frac{1}{s^{n-1}}dsdy) \\
&= \frac{-1}{n\alpha(n)}(\int_{B(0,r)-B(0,\epsilon)} f(y)\frac{1}{n-2}(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}})dy + \int_{B(0,\epsilon)} f(y)\frac{1}{n-2}(\frac{1}{\epsilon^{n-2}} - \frac{1}{r^{n-2}})dy) \quad by\,(3.2) \\
&= \frac{-1}{n(n-2)\alpha(n)}(\int_{B(0,r)} f(y)(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}})dy + \int_{B(0,\epsilon)} f(y)(\frac{1}{\epsilon^{n-2}} - \frac{1}{|y|^{n-2}})dy)
\end{aligned}$$

Suppose that we proved

$$\lim_{\epsilon\to 0} \int_{B(0,\epsilon)} f(y)(\frac{1}{\epsilon^{n-2}} - \frac{1}{|y|^{n-2}})dy = 0 \quad (3.3)$$

Then

$$\begin{aligned}
\phi(r) - u(0) &= \lim_{\epsilon\to 0} \phi(r) - \phi(\epsilon) \ by\ (3.1) \\
&= \lim_{\epsilon\to 0} \int_\epsilon^r \phi'(s)ds \\
&= \lim_{\epsilon\to 0} \frac{-1}{n(n-2)\alpha(n)}\left(\int_{B(0,r)} f(y)\left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}}\right)dy + \int_{B(0,\epsilon)} f(y)\left(\frac{1}{\epsilon^{n-2}} - \frac{1}{|y|^{n-2}}\right)dy\right) \\
&= \frac{-1}{n(n-2)\alpha(n)} \int_{B(0,r)} f(y)\left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}}\right)dy
\end{aligned}$$

Therefore

$$\begin{aligned}
u(0) &= \phi(r) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} f(y)\left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}}\right)dy \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(0,r)} u(y)dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} f(y)\left(\frac{1}{|y|^{n-2}} - \frac{1}{r^{n-2}}\right)dy
\end{aligned}$$

But $u = g$ on $\partial B(0,r)$, so the equality holds. So we only need to prove (3.3)

$$\lim_{\epsilon\to 0} \int_{B(0,\epsilon)} f(y)\left(\frac{1}{\epsilon^{n-2}} - \frac{1}{|y|^{n-2}}\right)dy = \lim_{\epsilon\to 0} \frac{1}{\epsilon^{n-2}} \int_{B(0,\epsilon)} f(y)dy - \lim_{\epsilon\to 0} \int_{B(0,\epsilon)} \frac{f(y)}{|y|^{n-2}}$$

But

$$\begin{aligned}
\lim_{\epsilon\to 0} \frac{1}{\epsilon^{n-2}} \int_{B(0,\epsilon)} f(y)dy &= \lim_{\epsilon\to 0} n\alpha(n)\epsilon^2 \frac{1}{m(B(0,\epsilon))} \int_{B(0,\epsilon)} f(y)dy \\
&= \lim_{\epsilon\to 0} n\alpha(n)\epsilon^2 f(0) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{B(0,\epsilon)} \frac{f(y)}{|y|^{n-2}} &= \lim_{\epsilon\to 0} \int_0^\epsilon \int_{\partial B(0,t)} \frac{f(y)}{|y|^{n-2}} dS(y)dt \\
&= \lim_{\epsilon\to 0} \int_0^\epsilon \frac{1}{t^{n-2}} \int_{\partial B(0,t)} f(y)dS(y)dt \\
&= \lim_{\epsilon\to 0} \int_0^\epsilon n\alpha(n)t \frac{1}{m(\partial B(0,t))} \int_{\partial B(0,t)} f(y)dS(y)dt \\
&= \lim_{\epsilon\to 0} \int_0^\epsilon n\alpha(n)tf(0)dt \\
&= \lim_{\epsilon\to 0} n\alpha(n)f(0)\frac{\epsilon^2}{2} \\
&= 0
\end{aligned}$$

Therefore

$$\lim_{\epsilon\to 0} \int_{B(0,\epsilon)} f(y)\left(\frac{1}{\epsilon^{n-2}} - \frac{1}{|y|^{n-2}}\right)dy = 0$$

so we are done.

4. First note that $\overline{U}$ and ∂U is compact, since U is bounded and open. So $max\, u = sup\, u$ on both $\overline{U}$ and ∂U. Define $u_\epsilon := u + \epsilon|x|^2$ for each $\epsilon > 0$. Then for $x \in U$, $\frac{\partial u_\epsilon}{\partial x_i} = u_{x_i} + 2\epsilon x_i$, so $\Delta u_\epsilon = \Delta u + 2n\epsilon = 2n\epsilon > 0$ since u is harmonic and $\epsilon > 0$. But $tr(Hess(u_\epsilon)) = \Delta u_\epsilon(x) > 0$, so $Hess(u_\epsilon)(x)$ is cannot be negative definite matrix. Therefore, x cannot be local maximum of the function u_ϵ, so u_ϵ cannot attain its max within U. Therefore, we conclude that

$$\max_{\overline{U}} u_\epsilon = \max_{\partial U} u_\epsilon$$

Since we know that U is bounded, we can assume $U \subset B(0,R)$ for some $R > 0$. Then

$$\underset{\overline{U}}{max}\, u \le \underset{\overline{U}}{max}\, u_\epsilon = \underset{\partial U}{max}\, u_\epsilon \le \underset{\partial U}{max}\, u + \underset{\partial U}{max}\, \epsilon|x|^2 \le \underset{\partial U}{max}\, u + \epsilon R^2$$

Let $\epsilon \to 0$, then $\underset{\overline{U}}{max}\, u \le \underset{\partial U}{max}\, u$. Since $\partial U \subset \overline{U}$, we conclude $\underset{\overline{U}}{max}\, u = \underset{\partial U}{max}\, u$

5. (a) As in the proof in the mean value property (or in the problem 3), define a function

$$\Phi(s) = \frac{1}{n\alpha(n)s^{n-1}} \int_{\partial B(x,s)} v(y) dS(y)$$

then the book proves

$$\Phi'(s) = \frac{1}{n\alpha(n)s^{n-1}} \int_{B(x,s)} \Delta v(y) dy$$

so $\Phi'(s) \ge 0$ since $\Delta v \ge 0$. Therefore $\Phi(r) \ge \Phi(\epsilon)$ for all $r > \epsilon > 0$. But $\lim_{\epsilon \to 0} v(\epsilon) = v(x)$ since Φ is an average integral of the function v.

(b) Assume that U is connected. Let $M := \max_{\overline{U}} u$ and $S = \{x \in U \big| v(x) = M\}$. If S is an empty set, then clearly $\max_{\overline{U}} \; = \max_{\partial U} u$, so we are done. Assume that S is not an empty set. We'll prove that S is both open and relatively closed, which will led us to say $S = U$ since U is connected. Let $a \in S$ and $\delta > 0$ such that $B(a, \delta) \subset U$. Then for all $\delta \ge \epsilon > 0$

$$v(a) = \frac{1}{n\alpha(n)s^{n-1}} \int_{\partial B(a,\delta)} v(y) dS(y) \le v(a)$$

therefore $v = v(a)$ within $B(a, \delta)$, ie $B(a, \delta) \subset S$, so S is open. Moreover S is relatively closed since u is a continuous function. Thus $S = U$, i.e. u is a constant function in U, and also in $\overline{U}$ since u is continuous. Therefore $\max_{\overline{U}} \; = \max_{\partial U} u$. Since this is true for all connected parts of the nonconnected open set U, we can say $\max_{\overline{U}} \; = \max_{\partial U} u$.

Second way
If U were bounded, we could use problem 4. Define $v_\epsilon(x) = v(x) + \epsilon|x|^2$. Then $\Delta v_\epsilon = \Delta v + 2n\epsilon > 0$, the rest are the same.

(c) We do straight calculation

$$\frac{\partial v}{\partial x_i} = \frac{\partial \phi(u)}{\partial x_i} = \phi'(u) u_{x_i}$$

So

$$\frac{\partial^2 v}{\partial x_i^2} = \phi''(u) u_{x_i}^2 + \phi'(u) u_{x_i x_i}$$

Therefore

$$\Delta v = \phi''(u) \big(\sum_{i=1}^n u_{x_i}^2 \big) + \phi'(u) \Delta u \ge 0$$

(d) Follow the calculations

$$v = |Du|^2 = \sum_{i=1}^n u_{x_i}^2$$

$$\frac{\partial v}{\partial x_k} = \sum_{i=1}^n 2 u_{x_i} u_{x_i x_k}$$

$$\frac{\partial^2 v}{\partial x_k^2} = 2\sum_{i=1}^{n} u_{x_i x_k}^2 + u_{x_i x_k x_k}$$

$$\Delta v = 2\sum_{i,k=1}^{n} u_{x_i x_k}^2 + \sum_{i=1}^{n} \Delta u_{x_i} \geq \sum_{i=1}^{n} \Delta u_{x_i}$$

Since u is harmonic, u_{x_i} is harmonic for all $i = 1, 2, ..., n$. Thus $\Delta v \geq \sum_{i=1}^{n} \Delta u_{x_i} = 0$

6. Since U is bounded, we can assume that $\overline{U} \subset B(0,R)$ for some fixed $R > 0$. Choose C such that $C > \max(1, \frac{R^2}{2n})$. Let's define $\lambda := \max_{\overline{U}} |f|$ (??? can we, can f be extended continuously over $\overline{U}$) Since $\Delta(u + \lambda\frac{|x|^2}{2n}) = \Delta u + n\frac{2\lambda}{2n} \geq 0$ $u + \lambda\frac{|x|^2}{2n}$ is subharmonic. By the problem 5

$$\begin{aligned} \max_{\overline{U}} (u + \lambda\frac{|x|^2}{2n}) &= \max_{\partial U} (u + \lambda\frac{|x|^2}{2n}) \\ &\leq \max_{\partial U} |u| + \lambda\frac{R^2}{2n} \\ &\leq \max_{\partial U} |g| + \frac{R^2}{2n}\max_{\overline{U}} |f| \\ &\leq C(\max_{\partial U} |g| + \max_{\overline{U}} |f|) \end{aligned}$$

7. We are assuming that u is harmonic in an open set U satisfying $\overline{B(0,r)} \subset U \subseteq \Re^n$. Let's write the Poisson's formula $x \in B(0,r)$,

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} dy$$

Note that by the triangle inequality

$$r + |x| = |y| + |x| \geq |x - y| \geq ||y| - |x|| = r - |x|$$

for $y \in \partial B(0,r)$. Moreover,

$$\int_{\partial B(0,r)} u(y)dy = n\alpha(n)r^{n-1}u(0)$$

by the mean value property of the harmonic functions. Then

$$r^{n-2}\frac{r - |x|}{(r+|x|)^{n-1}}u(0) = \frac{r^2 - |x|^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{u(y)}{(r+|x|)^n} dy \leq \frac{r^2 - |x|^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} dy = u(x)$$

Similarly

$$r^{n-2}\frac{r + |x|}{(r-|x|)^{n-1}}u(0) = \frac{r^2 - |x|^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{u(y)}{(r-|x|)^n} dy \geq \frac{r^2 - |x|^2}{n\alpha(n)} \int_{\partial B(0,r)} \frac{u(y)}{|x-y|^n} dy = u(x)$$

8. (i) We'll prove the Theorem 15 in the chapter 2.2. Take $u = 1$ in $B(0,r)$ and consequently $g = 1$ on $\partial B(0,r)$, then by theorem 12 (Representation Formula using Green's function), for any $x \in B(0,r)$

$$1 = u(x) = \int_{\partial B(0,r)} g(y)K(x,y)dS(y) = \int_{\partial B(0,r)} K(x,y)dS(y) \tag{8.1}$$

Let's back to the problem. First, we shall show that u_{x_i} exist and it equals to $\int_{\partial B(0,r)} g(y)K_{x_i}(x,y)dS(y)$. It's enough to show that

$$\lim_{h\to 0} \frac{u(x + he_i) - u(x)}{h} = \int_{\partial B(0,r)} g(y)K_{x_i}(x,y)dS(y)$$

Let's denote

$$v(x,h) := \frac{u(x+he_i)-u(x)}{h} = \int_{\partial B(0,r)} g(y)\Big(\frac{K(x+he_i,y)-K(x,y)}{h}\Big)dS(y)$$

By Mean Value Theorem, $K(x+he_i,y)-K(x,y) = hK_{x_i}(x+\gamma(h),y)$ for some $0<\gamma(h)<h$. Therefore

$$v(x,h) = \int_{\partial B(0,r)} g(y)K_{x_i}(x+\gamma(h),y)dS(y)$$

$$v(x,h) - \int_{\partial B(0,r)} g(y)K_{x_i}(x,y)dS(y) = \int_{\partial B(0,r)} g(y)\big(K_{x_i}(x+\gamma(h),y)-K_{x_i}(x,y)\big)dS(y)$$

Let $\mu := \operatorname{dist}(x,\partial B(0,r)) = \inf\{|x-y|\,\big|y\in\partial B(0,r)\}$ and consider the continuous function $K_{x_i}(x,y)$ on the compact set $\overline{U(0,r-\frac{\mu}{2})}\times\partial B(0,r)$. Thus $K_{x_i}(x,y)$ is uniformly continuous. Also, since g is continuous in the compact set $\partial B(0,r)$, g is also bounded. Assume $|g|<M$. Choose $\epsilon>0$. Since $K_{x_i}(x,y)$ is uniformly continuous, there exist $\delta>0$ such that $|(x_1,y_1)-(x_2,y_2)|_{\Re^{2n}}<\delta$ implies $|K_{x_i}(x_1,y_1)-K_{x_i}(x_2,y_2)|<\epsilon$ Thus, for $h<\min\{\mu,\delta\}$,

$$\begin{aligned}
\big|v(x,h) - \int_{\partial B(0,r)} g(y)K_{x_i}(x,y)dS(y)\big| &= \big|\int_{\partial B(0,r)} g(y)\big(K_{x_i}(x+\gamma(h),y)-K_{x_i}(x,y)\big)dS(y)\big| \\
&\leq \int_{\partial B(0,r)} |g(y)|\big|K_{x_i}(x+\gamma(h),y)-K_{x_i}(x,y)\big|dS(y) \\
&\leq \int_{\partial B(0,r)} |g(y)|\big|K_{x_i}(x+\gamma(h),y)-K_{x_i}(x,y)\big|dS(y) \\
&\leq \int_{\partial B(0,r)} |g(y)|\epsilon dS(y) \\
&\leq M\epsilon n\alpha(n)r^{n-1}
\end{aligned}$$

which goes 0 as ϵ goes 0. Therefore $\lim\limits_{h\to 0}\frac{u(x+he_i)-u(x)}{h}$ exist and it equals to $\int_{\partial B(0,r)} g(y)K_{x_i}(x,y)dS(y)$ In the same way, we can prove for all $\alpha=(\alpha_1,\alpha_2,...,\alpha_n)$,

$$D^\alpha u(x) = \int_{\partial B(0,r)} g(y)D^\alpha K(x,y)dS(y)$$

since K is smooth, so is u, i.e $u\in C^\infty(B(0,r))$

(ii) By (i), we can say $\Delta u = \int_{\partial B(0,r)} g(y)\Delta_x K(x,y)dS(y)$. But $\Delta_x K(x,y)=0$ within $B(0,r)$ (I may add calculation) so $\Delta u = 0$

(iii) We'll follow the similar path as in the proof of the **theorem 14**. Let $\epsilon>0$, since g is continuous, choose $\delta>0$ such that $|y-x^0|<\delta$ implies $|g(y)-g(x^0)|<\epsilon$. Moreover, since g is defined in the compact set $\partial B(0,r)$, it is bounded, assume $|g|<M$. By (8.1),

$$\begin{aligned}
|u(x)-g(x^0)| &= \big|\int_{\partial B(0,r)} g(y)K(x,y)dS(y) - \int_{\partial B(0,r)} g(x^0)K(x,y)dS(y)\big| \\
&= \big|\int_{\partial B(0,r)} (g(y)-g(x^0))K(x,y)dS(y)\big| \\
&\leq \int_{\partial B(0,r)} |g(y)-g(x^0)|K(x,y)dS(y) \\
&= \int_{\partial B(0,r)\cap B(x^0,\delta)} |g(y)-g(x^0)|K(x,y)dS(y) + \\
&\quad \int_{\partial B(0,r)-B(x^0,\delta)} |g(y)-g(x^0)|K(x,y)dS(y) \\
&= I+J
\end{aligned}$$

Clearly $I < \int_{\partial B(0,r)\cap B(x^0,\delta)} \epsilon K(x,y)dS(y) < \epsilon \int_{\partial B(0,r)} K(x,y)dS(y) = \epsilon$ also by (8.1). Furthermore, if $|x-x^0| < \frac{\delta}{2}$ and $|y-x^0| > \delta$, we have

$$|y-x^0| < |y-x| + |x-x^0| < |y-x| + \frac{\delta}{2} < |y-x| + \frac{1}{2}|y-x|$$

therefore $|y-x| > \frac{1}{2}|y-x^0|$, so

$$K(x,y) = \frac{r^2-|x|^2}{n\alpha(n)|x-y|^n} < \frac{2^n(r^2-|x|^2)}{n\alpha(n)}\frac{1}{|y-x^0|^n} < \frac{2^n(r^2-|x|^2)}{n\alpha(n)\delta^n}$$

for $|x-x^0| < \frac{\delta}{2}$ and $y \in \partial B(0,r) - B(x^0,\delta)$. By the triangle inequality, we have $|g(y)-g(x^0)| < 2M$. So finally

$$\begin{aligned} J &< 2M\int_{\partial B(0,r)-B(x^0,\delta)} \frac{2^n(r^2-|x|^2)}{n\alpha(n)\delta^n}dS(y) \\ &\leq 2M\frac{2^n(r^2-|x|^2)}{n\alpha(n)\delta^n}n\alpha(n)r^{n-1} \\ &= (r^2-|x|^2)\frac{2^{n+1}Mr^{n-1}}{\delta^n} \end{aligned}$$

which goes 0 as x goes x^0 since $x^0 \in \partial B(0,r)$ so $|x| \to |x^0| = r$

9. By Poisson's formula for the half space, for $x = (x_1, x_2, ..., x_n) \in \Re^n_+$

$$u(x) = \frac{2x_n}{n\alpha(n)}\int_{\partial\Re^n_+} \frac{g(y)}{|x-y|^n}dS(y) = \frac{2x_n}{n\alpha(n)}\int_{\Re^{n-1}} \frac{\tilde{g}(\tilde{y})}{|x-y|^n}d\tilde{y}$$

where $\tilde{y} = (y_1, y_2, ..., y_{n-1}) \in \Re^{n-1}$ for $y = (y_1, y_2, ..., y_{n-1}, 0) \in \partial\Re^n_+$. Let's put $x = \lambda e_n = (0, 0, ..., \lambda)$, then $|x-y|^2 = |\tilde{y}|^2 + \lambda^2$, therefore the equality becomes

$$u(\lambda e_n) = \frac{2}{n\alpha(n)}\int_{\Re^{n-1}} \tilde{g}(\tilde{y})\frac{\lambda}{(|\tilde{y}|^2+\lambda^2)^{\frac{n}{2}+1}}d\tilde{y}$$

We'll compute the partial derivative of u WRT x_n at $x = \lambda e_n$.

$$\begin{aligned} u_{x_n}(\lambda e_n) &= \frac{\partial}{\partial\lambda}u(\lambda e_n) \\ &= \frac{\partial}{\partial\lambda}\frac{2}{n\alpha(n)}\int_{\Re^{n-1}} \tilde{g}(\tilde{y})\frac{\lambda}{(|\tilde{y}|^2+\lambda^2)^{\frac{n}{2}+1}}d\tilde{y} \qquad (9.0) \\ &= \frac{2}{n\alpha(n)}\int_{\Re^{n-1}} \frac{\partial}{\partial\lambda}\tilde{g}(\tilde{y})\frac{\lambda}{(|\tilde{y}|^2+\lambda^2)^{\frac{n}{2}+1}}d\tilde{y} \\ &= \frac{2}{n\alpha(n)}\int_{\Re^{n-1}} g(\tilde{y})\frac{|\tilde{y}|^2-(n-1)\lambda^2}{|\tilde{y}^2+\lambda^2|^{\frac{n}{2}+1}} \\ &= \frac{2}{n\alpha(n)}\int_0^\infty\int_{\partial B^{n-1}(0,r)} g(\tilde{y})\frac{|\tilde{y}|^2-(n-1)\lambda^2}{(|\tilde{y}|^2+\lambda^2)^{\frac{n}{2}+1}}dS(\tilde{y})dr \quad (9.1) \\ &= \frac{2}{n\alpha(n)}\int_0^\infty\int_{\partial B^{n-1}(0,r)} g(\tilde{y})\frac{r^2-(n-1)\lambda^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}}dS(\tilde{y})dr \end{aligned}$$

At (9.0), we used the fact in the proof of the **theorem 14**, which is $u_{x_i}(x) = \int_{\partial\Re^n_0} g(y)K_{x_i}(x,y)dy$ At 9.1, we used polar coordinates. $B^{n-1}(0,r)$ denotes the open ball centered at origin with radious r in $\Re^{n-1}$. Define functions

$$f_1(r,\lambda) = \int_{\partial B^{n-1}(0,r)} g(\tilde{y})\frac{r^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}}dS(\tilde{y})$$

and

$$f_2(r,\lambda) = \int_{\partial B^{n-1}(0,r)} g(\tilde{y}) \frac{-(n-1)\lambda^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}} dS(\tilde{y})$$

Note that

$$u_{x_n}(\lambda e_n) = \int_1^\infty f_1(r,\lambda)dr + \int_0^1 f_1(r,\lambda)dr + \int_1^\infty f_2(r,\lambda)dr + \int_0^1 f_2(r,\lambda)dr$$

Assume that we know $\int_1^\infty f_1(r,\lambda)dr$, $\int_1^\infty f_2(r,\lambda)dr$ and $\int_0^1 f_2(r,\lambda)dr$ are bounded and $\lim_{\lambda\to 0} \int_0^1 f_1(r,\lambda)dr = \infty$. Then we can conclude that $u_{x_n}(\lambda e_n) \to \infty$ as $\lambda \to 0$; therefore $Du(\lambda e_n)$ is not bounded near 0. Let's prove the assumptions. Since we know g is bounded, we assume $|g| < M$

$$\begin{aligned}
\left|\int_1^\infty f_1(r,\lambda)dr\right| &\leq \int_1^\infty \int_{\partial B^{n-1}(0,r)} \left|g(\tilde{y})\frac{r^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}}\right| dS(\tilde{y})dr \\
&\leq M\int_1^\infty \int_{\partial B^{n-1}(0,r)} \frac{r^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}} dS(\tilde{y})dr \\
&= M\int_1^\infty (n-1)\alpha(n-1)r^{n-2}\frac{r^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}} dS(\tilde{y})dr \\
&= M(n-1)\alpha(n-1)\int_1^\infty \frac{r^n}{(r^2+\lambda^2)^{\frac{n}{2}+1}}dr \qquad (9.2) \\
&\leq M(n-1)\alpha(n-1)\int_1^\infty \frac{r}{(r^2+\lambda^2)^{\frac{3}{2}}}dr \\
&= M(n-1)\alpha(n-1)\left(\frac{-1}{(r^2+\lambda^2)^{\frac{1}{2}}}\Big|_1^\infty\right) \\
&= M(n-1)\alpha(n-1)\frac{1}{\lambda^2+1} \\
&\leq M(n-1)\alpha(n-1)
\end{aligned}$$

Note that at (9.2), we used the inequality

$$\frac{r^n}{(r^2+\lambda^2)^{\frac{n}{2}+1}} = \frac{r}{(r^2+\lambda^2)^{\frac{3}{2}}}\left(\frac{r}{(r^2+\lambda^2)^{\frac{1}{2}}}\right)^{n-1} \leq \frac{r}{(r^2+\lambda^2)^{\frac{3}{2}}}$$

We find that $\int_1^\infty f_1(r,\lambda)dr$ is bounded. For the second assumption

$$\begin{aligned}
\left|\int_1^\infty f_2(r,\lambda)dr\right| &\leq \int_1^\infty \int_{\partial B^{n-1}(0,r)} \left|g(\tilde{y})\frac{-(n-1)\lambda^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}}\right| dS(\tilde{y})dr \\
&\leq M(n-1)\lambda^2\int_1^\infty \int_{\partial B^{n-1}(0,r)} \frac{1}{(r^2+\lambda^2)^{\frac{n}{2}+1}} dS(\tilde{y})dr \\
&= M(n-1)^2\alpha(n-1)\lambda^2\int_1^\infty \frac{r^{n-2}}{(r^2+\lambda^2)^{\frac{n}{2}+1}}dr \qquad (9.3) \\
&\leq M(n-1)^2\alpha(n-1)\lambda^2\int_1^\infty \frac{r}{(r^2+\lambda^2)^{\frac{3}{2}}}dr \\
&= M(n-1)^2\alpha(n-1)\lambda^2\left(\frac{-1}{(r^2+\lambda^2)^{\frac{1}{2}}}\Big|_1^\infty\right) \\
&= M(n-1)^2\alpha(n-1)\frac{\lambda^2}{\lambda^2+1} \\
&\leq M(n-1)^2\alpha(n-1)
\end{aligned}$$

At (9.3) we used the inequality

$$\frac{r^{n-2}}{(r^2+\lambda^2)^{\frac{n}{2}+1}} = \frac{r}{(r^2+\lambda^2)^{\frac{3}{2}}}\left(\frac{r}{(r^2+\lambda^2)^{\frac{1}{2}}}\right)^{n-1}\frac{1}{r^2} \leq \frac{r}{(r^2+\lambda^2)^{\frac{3}{2}}}$$

The inequality is true for $r > 1$, which is enough since we integrate from 1 to ∞. For third assumption, realize that for $r \leq 1$

$$f_2(r) = \int_{\partial B^{n-1}(0,r)} |\tilde{y}| \frac{-(n-1)\lambda^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}} dS(\tilde{y}) = -\lambda^2(n-1)^2\alpha(n-1)\frac{r^{n-1}}{(r^2+\lambda^2)^{\frac{n}{2}+1}}$$

Since $g(\tilde{y}) = |\tilde{y}|$ for $|\tilde{y}| \leq 1$. Therefore

$$\begin{aligned}
\left|\int_1^\infty f_2(r,\lambda)dr\right| &\leq \lambda^2(n-1)^2\alpha(n-1)\int_0^1 \frac{r^{n-1}}{(r^2+\lambda^2)^{\frac{n}{2}+1}} \\
&\leq \lambda^2(n-1)^2\alpha(n-1)\int_0^1 \frac{r}{(r^2+\lambda^2)^2} \qquad (9.4) \\
&\leq \lambda^2(n-1)^2\alpha(n-1)\left(\frac{-1}{2(r^2+\lambda^2)}\bigg|_0^{r=1}\right) \\
&= (n-1)^2\alpha(n-1)\frac{1}{2(\lambda^2+1)} \\
&\leq \frac{1}{2}(n-1)^2\alpha(n-1)
\end{aligned}$$

At (9.4), we used the inequality

$$\frac{r^{n-1}}{(r^2+\lambda^2)^{\frac{n}{2}+1}} = \frac{r}{(r^2+\lambda^2)^2}\left(\frac{r}{(r^2+\lambda^2)^{\frac{1}{2}}}\right)^{n-2} \leq \frac{r}{(r^2+\lambda^2)^2}$$

since $n > 1$.

Let's prove the last assumption. For $r < 1$

$$\begin{aligned}
f_1(r,\lambda) &= \int_{\partial B^{n-1}(0,r)} g(\tilde{y})\frac{r^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}} dS(\tilde{y}) \\
&= \int_{\partial B^{n-1}(0,r)} |\tilde{y}|\frac{r^2}{(r^2+\lambda^2)^{\frac{n}{2}+1}} dS(\tilde{y}) \\
&= \int_{\partial B^{n-1}(0,r)} \frac{r^3}{(r^2+\lambda^2)^{\frac{n}{2}+1}} dS(\tilde{y}) \\
&= (n-1)\alpha(n-1)r^{n-2}\frac{r^3}{(r^2+\lambda^2)^{\frac{n}{2}+1}} \\
&= (n-1)\alpha(n-1)\frac{r^{n+1}}{(r^2+\lambda^2)^{\frac{n}{2}+1}}
\end{aligned}$$

since $g(\tilde{y}) = |\tilde{y}|$ for $|\tilde{y}| \leq 1$. Let $(n-1)\alpha(n-1) = \mu$. Then

$$\begin{aligned}
\int_0^1 f_1(r,\lambda)dr &= \mu\int_0^1 \frac{r^{n+1}}{(r^2+\lambda^2)^{\frac{n}{2}+1}}dr \\
&\geq \mu\int_\lambda^1 \frac{r^{n+1}}{(r^2+\lambda^2)^{\frac{n}{2}+1}}dr \qquad (9.5) \\
&\geq \mu\int_\lambda^1 \frac{1}{2^{\frac{n}{2}+1}}\frac{1}{r}dr \\
&= \mu\frac{1}{2^{\frac{n}{2}+1}}\int_\lambda^1 \frac{1}{r}dr \\
&= \frac{\mu}{2^{\frac{n+2}{2}}}\left(log(r)\bigg|_\lambda^1\right) \\
&= \frac{\mu}{2^{\frac{n+2}{2}}}(-log(\lambda))
\end{aligned}$$

which goes infinity as $\lambda \to 0$. Note that at (9.5), we used inequality

$$\frac{r^{n+1}}{(r^2+\lambda^2)^{\frac{n}{2}+1}} \geq \frac{1}{2^{\frac{n}{2}+1}}\frac{1}{r}$$

which is equivalent

$$(2r^2)^{\frac{n}{2}+1} \geq (r^2+\lambda^2)^{\frac{n}{2}+1}$$

which is true for $r \geq \lambda$, so we are done.

11. First, let's focus on the function $\tilde{x} : \Re^n \to \Re^n$ which takes x to $\frac{\tilde{x}}{|x|^2}$.

$$\tilde{x} = \tilde{x}(x) = (x_1|x|^{-2}, x_2|x|^{-2}, ..., x_n|x|^{-2})$$

and

$$D(\tilde{x}) = D_x(\tilde{x}) = \begin{bmatrix} (|x|^2-2x_1^2)|x|^{-4} & -2x_1x_2|x|^{-4} & -2x_1x_3|x|^{-4} & \cdots & -2x_1x_n|x|^{-4} \\ -2x_2x_1|x|^{-4} & (|x|^2-2x_2^2)|x|^{-4} & -2x_2x_3|x|^{-4} & \cdots & -2x_2x_n|x|^{-4} \\ -2x_3x_1|x|^{-4} & -2x_3x_2|x|^{-4} & (|x|^2-2x_3^2)|x|^{-4} & \cdots & -2x_3x_n|x|^{-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -2x_nx_1|x|^{-4} & -2x_nx_2|x|^{-4} & -2x_nx_3|x|^{-4} & \cdots & (|x|^2-2x_n^2)|x|^{-4} \end{bmatrix}$$

Let's denote $D(\tilde{x}) = (\alpha_j^i(x))$ where

$$\alpha_j^i(x) = \big(D(\tilde{x})\big)_{ij} = \frac{\partial}{\partial x_j}\left(\frac{x_i}{|x|^2}\right) = \begin{cases} -2x_ix_j|x|^{-4} & \text{if } j \neq i \\ (|x|^2-2x_i^2)|x|^{-4} & \text{if } j = i \end{cases}$$

Clearly $\alpha_j^i = \alpha_i^j$, ie the matrix $D(\tilde{x})$ is symmetric. Let $r_i(x) = (\alpha_i^1(x), \alpha_i^2(x), ..., \alpha_i^n(x)) = (\alpha_1^i(x), \alpha_2^i(x), ..., \alpha_n^i(x)$ denote the i^{th} row and column. Note that $r_i(x) = D\big(\frac{x_i}{|x|^2}\big)$.

Lemma 1. $D(\tilde{x}) \cdot D(\tilde{x}) = |x|^{-4} \cdot I$
Proof We need to prove that

$$r_i(x) \cdot r_j(x) = \begin{cases} |x|^{-4} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

For $j = i$

$$\begin{aligned} r_i(x) \cdot r_i(x) &= \sum_{k=1}^{n} (\alpha_k^i(x))^2 \\ &= ((|x|^2-2x_i^2)|x|^{-4})^2 + \sum_{k=1,k\neq i}^{n} (-2x_ix_k|x|^{-4})^2 \\ &= (|x|^4 - 4|x|^2x_i^2 + 4x_i^4)|x|^{-8} + \sum_{k=1,k\neq i}^{n} 4x_i^2x_k^2|x|^{-8} \\ &= |x|^{-8}\left(|x|^4 - 4|x|^2x_i^2 + 4x_i^2\sum_{k=1}^{n} x_k^2\right) \\ &= |x|^{-4} \end{aligned}$$

For $j \neq i$

$$\begin{aligned}
r_i(x) \cdot r_j(x) &= \sum_{k=1}^{n} \alpha_k^i(x)\alpha_k^j(x) \\
&= -2x_jx_i|x|^{-4}(|x|^2 - 2x_i^2)|x|^{-4} - 2x_jx_i|x|^{-4}(|x|^2 - 2x_j^2)|x|^{-4} + \sum_{k=1, k\neq i,j}^{n} 4x_ix_jx_k^2|x|^{-8} \\
&= 2x_ix_j|x|^{-8}\bigg(2x_i^2 + 2x_j^2 - 2|x|^2 + \sum_{k=1, k\neq i,j}^{n} 2x_k^2\bigg) \\
&= 2x_ix_j|x|^{-8}\bigg(-2|x|^2 + \sum_{k=1}^{n} 2x_k^2\bigg) \\
&= 0
\end{aligned}$$

So the lemma is proved.

Corollary (1)

$$r_i(x) \cdot r_j(x) = \begin{cases} |x|^{-4} & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

Let's continue the problem. By product rule

$$D(\tilde{u}(x)) = D(u(\tilde{x})|x|^{2-n}) = Du(\tilde{x}) \cdot D(\tilde{x})|x|^{2-n} + u(\tilde{x})D(|x|^{2-n})$$

So

$$\frac{\partial}{\partial x_i}\tilde{u}(x) = Du(\tilde{x}) \cdot r_i(x)|x|^{2-n} + u(\tilde{x})(2-n)x_i|x|^{-n}$$

$$\frac{\partial^2}{\partial x_i^2}\tilde{u}(x) = \frac{\partial}{\partial x_i}(Du(\tilde{x})) \cdot r_i(x)|x|^{2-n} + Du(\tilde{x}) \cdot \bigg(\frac{\partial}{\partial x_i} r_i(x)\bigg)|x|^{2-n} + (2-n)Du(\tilde{x}) \cdot r_i(x)x_i|x|^{-n} +$$

$$+(2-n)\bigg(\frac{\partial}{\partial x_i}u(\tilde{x})\bigg)x_i|x|^{-n} + (2-n)u(\tilde{x})|x|^{-n} + n(n-2)u(\tilde{x})x_i^2|x|^{-n-2}$$

We shall prove that

$$\textbf{(1)} \qquad \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(Du(\tilde{x})) \cdot r_i(x)|x|^{2-n} = 0$$

$$\textbf{(2)} \qquad \sum_{i=1}^{n} Du(\tilde{x}) \cdot \bigg(\frac{\partial}{\partial x_i} r_i(x)\bigg)|x|^{2-n} = -2(n-2)|x|^{-2-n}Du(\tilde{x}) \cdot x$$

$$\textbf{(3)} \qquad \sum_{i=1}^{n}(2-n)Du(\tilde{x}) \cdot r_i(x)x_i|x|^{-n} = (n-2)|x|^{-2-n}Du(\tilde{x}) \cdot x$$

$$\textbf{(4)} \qquad \sum_{i=1}^{n}(2-n)\bigg(\frac{\partial}{\partial x_i}u(\tilde{x})\bigg)x_i|x|^{-n} = (n-2)|x|^{-2-n}Du(\tilde{x}) \cdot x$$

$$\textbf{(5)} \qquad \sum_{i=1}^{n} n(n-2)u(\tilde{x})x_i^2|x|^{-n-2} = n(n-2)u(\tilde{x})|x|^{-n}$$

Note that **(2)**+**(3)**+**(4)**=0 and **(5)**=$n(n-2)u(\tilde{x})|x|^{-n} = -\sum_{i=1}^{n}(2-n)u(\tilde{x})|x|^{-n}$, therefore if **(1)-(5)** are all true, we can conclude that $\Delta\tilde{u} = 0$. Let's start to prove claims

Proof (1) We shall prove that $\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(Du(\tilde{x})) \cdot r_i(x) = 0$, then we can immediately conclude that

$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(Du(\tilde{x})) \cdot r_i(x)|x|^{2-n} = 0$. First, let's calculate $\frac{\partial}{\partial x_i}(Du(\tilde{x})) = \frac{\partial}{\partial x_i}(u_{x_1}(\tilde{x}), u_{x_2}(\tilde{x}), ..., u_{x_n}(\tilde{x}))$. For $j = 1, 2, 3, ..., n$,

$$\frac{\partial}{\partial x_i}\left(u_{x_j}(\tilde{x})\right) = Du_{x_j}(\tilde{x}) \cdot r_i(x) = (u_{x_j x_1}(\tilde{x}), u_{x_j x_2}(\tilde{x}), ..., u_{x_j x_n}(\tilde{x})) \cdot r_i(x) = \sum_{k=1}^{n} u_{x_j x_k} \alpha_k^i(x)$$

So

$$\frac{\partial}{\partial x_i}(Du(\tilde{x})) \cdot r_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{n} u_{x_j x_k} \alpha_k^i(x) \alpha_j^i(x)$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(Du(\tilde{x})) \cdot r_i(x) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} u_{x_j x_k}(\tilde{x}) \alpha_k^i(x) \alpha_j^i(x) \\
&= \sum_{j,k=1}^{n} u_{x_j x_k}(\tilde{x}) \sum_{i=1}^{n} \alpha_k^i(x) \alpha_j^i(x) \\
&= \sum_{j,k=1}^{n} u_{x_j x_k}(\tilde{x}) r_j(x) r_i(x) \qquad (11.1) \\
&= \sum_{j=1}^{n} u_{x_j x_j}(\tilde{x}) |x|^{-4} \\
&= |x|^{-4} \Delta u(\tilde{x}) \\
&= 0
\end{aligned}$$

since u is harmonic. At (11.1), we used **corollary 1**

Proof (2) Let's start by calculating $\frac{\partial}{\partial x_i} r_i(x)$. For $j \neq i$

$$\frac{\partial}{\partial x_i} \alpha_j^i(x) = \frac{\partial}{\partial x_i}(-2x_i x_j |x|^{-4}) = -2x_j |x|^{-4} + 8x_j x_i^2 |x|^{-6}$$

and for $j = i$

$$\frac{\partial}{\partial x_i} \alpha_i^i = \frac{\partial}{\partial x_i}(|x|^{-2} - 2x_i^2 |x|^{-4}) = -6x_i |x|^{-4} + 8x_i^3 |x|^{-6}$$

So

$$\begin{aligned}
\sum_{i=1}^{n} Du(\tilde{x}) \cdot \left(\frac{\partial}{\partial x_i} r_i(x)\right) &= \sum_{i=1}^{n} \sum_{j=1}^{n} u_{x_j}(\tilde{x}) \frac{\partial}{\partial x_i} \alpha_j^i(x) \\
&= \sum_{j=1}^{n} u_{x_j}(\tilde{x}) \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \alpha_j^i(x) \\
&= \sum_{j=1}^{n} u_{x_j}(\tilde{x}) \Big(- 6x_j |x|^{-4} + 8x_j^3 |x|^{-6} + \sum_{i=1, i \neq j}^{n} -2x_j |x|^{-4} + 8x_j x_i^2 |x|^{-6} \Big) \\
&= \sum_{j=1}^{n} u_{x_j}(\tilde{x}) \Big((-2n-4) x_j |x|^{-4} + 8x_j |x|^{-6} \sum_{i=1}^{n} x_i^2 \Big) \\
&= \sum_{j=1}^{n} u_{x_j}(\tilde{x}) \big((-2n-4) x_j |x|^{-4} + 8x_j |x|^{-4} \big) \\
&= -2(n-2)|x|^{-4} \sum_{j=1}^{n} u_{x_j}(\tilde{x}) \cdot x_j \\
&= -2(n-2)|x|^{-4} Du(\tilde{x}) \cdot x
\end{aligned}$$

Finally

$$\sum_{i=1}^{n} Du(\tilde{x}) \cdot \left(\frac{\partial}{\partial x_i} r_i(x)\right)|x|^{2-n} = -2(n-2)|x|^{-4} Du(\tilde{x}) \cdot x|x|^{2-n} = -2(n-2)|x|^{-n-2} Du(\tilde{x}) \cdot x$$

so we proved **(2)**.

Proof (3) We do straight calculation

$$\begin{aligned}
\sum_{i=1}^{n} Du(\tilde{x}) \cdot r_i(x) x_i &= \sum_{i=1}^{n}\sum_{j=1}^{n} u_{x_j}(\tilde{x}) \alpha_j^i(x) x_i \\
&= \sum_{j=1}^{n} \left(u_{x_j}(\tilde{x}) \sum_{i=1}^{n} \alpha_j^i(x) x_i \right) \\
&= \sum_{j=1}^{n} \left(u_{x_j}(\tilde{x}) \left(x_j|x|^{-2} - 2x_j^3|x|^{-4} + \sum_{i=1, i\neq j}^{n} -2x_i x_j^2 |x|^{-4} \right)\right) \\
&= \sum_{j=1}^{n} \left(u_{x_j}(\tilde{x}) \left(x_j|x|^{-2} - 2x_j \sum_{i=1}^{n} x_i^2 |x|^{-4} \right)\right) \\
&= -\sum_{j=1}^{n} u_{x_j}(\tilde{x}) x_j |x|^{-2} \\
&= -|x|^{-2} Du(\tilde{x}) \cdot x
\end{aligned}$$

Finally

$$\begin{aligned}
\sum_{i=1}^{n} (2-n) Du(\tilde{x}) \cdot r_i(x) x_i |x|^{-n} &= (2-n)|x|^{-n}(-|x|^{-2} Du(\tilde{x}) \cdot x \\
&= (n-2)|x|^{-2-n} Du(\tilde{x}) \cdot x
\end{aligned}$$

Proof (4) Since $\frac{\partial}{\partial x_i} u_{\tilde{x}} = Du(\tilde{x}) = Du(\tilde{x}) r_i(x)$, we can immediately see **(4)=(3)**, so we are done.

Proof (5) This is also trivial, as follows

$$\begin{aligned}
\sum_{i=1}^{n} n(n-2) u(\tilde{x}) x_i^2 |x|^{-n-2} &= n(n-2) u(\tilde{x}) |x|^{-n-2} \sum_{i=1}^{n} x_i^2 \\
&= n(n-2) u(\tilde{x}) |x|^{-n-2} |x|^2 \\
&= n(n-2) u(\tilde{x}) |x|^{-n}
\end{aligned}$$

12. Let's use the notation $u(x, t; \lambda) := u_\lambda(x, t) = u(\lambda x, \lambda^2 t)$.
(a) We do straight calculation

$$\frac{\partial}{\partial t} u(x, t; \lambda) = \frac{\partial}{\partial t} u(\lambda x, \lambda^2 t) = \lambda^2 u_t(\lambda x, \lambda^2 t)$$

$$\frac{\partial}{\partial x_i} u(x, t; \lambda) = \frac{\partial}{\partial x_i} u(\lambda x, \lambda^2 t = \lambda u_{x_i}(\lambda x, \lambda^2 t)$$

$$\frac{\partial^2}{\partial x_i^2} u(x, t; \lambda) = \frac{\partial}{\partial x_i} \lambda u_{x_i}(\lambda x, \lambda^2 t) = \lambda^2 u_{x_i x_i} u(\lambda x, \lambda^2 t)$$

Therefore

$$u_t(x,t;\lambda) - \Delta_x u(x,t;\lambda) = \lambda^2 u_t(\lambda x, \lambda^2 t) - \lambda^2 \Delta_x u(\lambda x, \lambda^2 t) = 0$$

so $u_\lambda(x,t) = u(x,t;\lambda)$ solves heat equation.

(b)

$$\frac{\partial}{\partial \lambda} u(x,t;\lambda) = \frac{\partial}{\partial \lambda} u(\lambda x, \lambda^2 t) = x \cdot D_x u(\lambda x, \lambda^2 t) + 2\lambda t u_t(\lambda x, \lambda^2 t)$$

Since $u(x,t)$ is smooth, so is $u(x,t;\lambda)$. Therefore the derivatives commutes. More precisely,

$$(\frac{\partial}{\partial t} - \Delta_x)(\frac{\partial}{\partial \lambda} u(x,t;\lambda)) = \frac{\partial}{\partial \lambda}\Big((\frac{\partial}{\partial t} - \Delta_x) u(x,t;\lambda)\Big) = 0$$

by **(a)**. So the function $\frac{\partial}{\partial \lambda} u(x,t;\lambda)$ solves the heat equation for fixed λ. Take $\lambda = 1$, then

$$\frac{\partial}{\partial \lambda} u(x,t;\lambda) = \frac{\partial}{\partial \lambda} u(x,t;1) = x \cdot D_x u(x,t) + 2t u_t(x,t)$$

solves the heat equation.

13. **(a)** Let $z = z(x,t) = \frac{x}{\sqrt{t}}$. Then

$$u_t(x,t) = \frac{\partial}{\partial t} v(\frac{x}{\sqrt{t}}) = -\frac{1}{2}\frac{x}{t^{\frac{3}{2}}} v'(z(x,t))$$

$$u_x(x,t) = \frac{\partial}{\partial x} v(\frac{x}{\sqrt{t}}) = \frac{1}{\sqrt{t}} v'(z(x,t))$$

$$u_x x(x,t) = \frac{\partial}{\partial x}\frac{1}{\sqrt{t}} v'(\frac{x}{\sqrt{t}}) = \frac{1}{t} v''(z(x,t))$$

Therefore

$$u_t = u_x x$$

$$\Longleftrightarrow$$

$$-\frac{1}{2}\frac{x}{t^{\frac{3}{2}}} v'(z(x,t)) = \frac{1}{t} v''(z(x,t))$$

$$\Longleftrightarrow$$

$$-\frac{1}{2} z(x,t) v'(z(x,t)) = v''(z(x,t))$$

$$\Longleftrightarrow$$

$$\textbf{(*)} \qquad v''(z) + \frac{z}{2} v'(z) = 0$$

By **(*)**, v' satisfies $\frac{v''}{v'} = -\frac{z}{2}$, so

$$\log(v') = \int \frac{v''}{v'} + C_1 = \int \frac{-z}{2} + C_1 = -\frac{z^2}{4} + C_1$$

$$v'(z) = e^{-z^2/4 + C_1} = c e^{-z^2/4}$$

where $c = e^{C_1}$ is a constant. Thus

$$v(z) = \int v'(s) ds + d = c \int e^{-s^2/4} ds + d$$

14. Define function $v(x,t) = u(x,t)e^{ct}$. Then

$$v_t - \Delta v = u_t e^{ct} + uce^{ct} - e^{ct}\Delta u = e^{ct}(u_t - \Delta u + cu) = e^{ct} f \qquad (14.1)$$

and clearly $v = g$ on $\Re^n \times \{t = 0\}$. Using THEOREM 2 in section 2.3.1, the function

$$v(x,t) = \int_{\Re^n} \Phi(x-y,t)g(y)dy + \int_0^t \int_{\Re^n} \Phi(x-y,t-s)f(y,s)e^{cs}dyds$$

solves the equation (14.1). Thus the function $u(x,t) = v(x,t)e^{-ct}$ solves the system of equation.

16. Define $u_\epsilon(x,t) = u(x,t) - \epsilon t$ for each $\epsilon > 0$. We will prove first that u_ϵ attains it's maximum on Γ_T. For sake of contradiction, assume that u_ϵ has maximum at $(x^0,t^0) \in U_T = U \times (0,T]$. Define $v : \overline{U} \to \Re$ with $v(x) = u_\epsilon(x,t^0)$. Then v has maximum at $x^0 \in U$. Since $u \in C_1^2(U_T) \cap C(\overline{U_T})$, we have $v \in C_1^2(U) \cap C(\overline{U})$. Thus we must have $Hess(v)(x^0)$ is a negative definite matrix. So we must have

$$\Delta v(x^0) = tr(Hess(v)(x^0)) < 0 \qquad (16.1)$$

We'll consider two cases.

Case 1: $t^0 < T$
Since $(x^0,t^0) \in U \times (0,T)$ is global maximum, we have $\frac{\partial}{\partial t}u_\epsilon(x^0,t^0) = 0$. So

$$0 = \frac{\partial}{\partial t}u_\epsilon(x^0,t^0) = u_t(x^0,t^0) - \epsilon$$

Therefore we have

$$u_t(x^0,t^0) = \epsilon \qquad (16.2)$$

But

$$\begin{aligned} \Delta v(x^0) &= \Delta_x u_\epsilon(x^0,t^0) \\ &= \Delta u(x^0,t^0) \\ &= u_t(x^0,t^0) \qquad \text{by (16.2)} \\ &= \epsilon \\ &> 0 \end{aligned}$$

using (16.1), we get a contradiction.

Case 2: $t^0 = T$
Since $u_\epsilon(x^0,t^0)$ is global maximum, there exist $0 < T' < T$ such that $\frac{\partial}{\partial t}u_\epsilon(x^0,t) \geq 0$ for all $t \in (T',T)$. Since u solve heat equation

$$\begin{aligned} 0 &\leq \frac{\partial}{\partial t}u_\epsilon(x^0,t) \\ &= u_t(x^0,t) - \epsilon \\ &= \Delta u(x^0,t) - \epsilon \end{aligned}$$

so

$$\Delta u(x^0,t) \geq \epsilon \qquad (16.3)$$

for all $t \in (T',T)$. Also we know that $\Delta v(x^0) = \Delta_x u_\epsilon(x^0,T) = \Delta_x u(x^0,T)$. Since $u \in C_1^2(U_T) \cap C(\overline{U_T})$, $\Delta_x u$ is continuous. Using (16.3), we conclude that $\Delta v(x^0) \geq \epsilon > 0$, which is a contradiction with (16.1).

Corollary: For all $\epsilon > 0$

$$\max_{\overline{U_T}} u_\epsilon = \max_{\Gamma_T} u_\epsilon$$

By using the **corollary**, let's prove that u also attains its maximum on Γ_T. Since $u = u_\epsilon + \epsilon t$

$$\begin{aligned}
\max_{\overline{U}_T} u &\leq \max_{\overline{U}_T} (u_\epsilon + \epsilon t) \\
&\leq \max_{\overline{U}_T} u_\epsilon + \max_{\overline{U}_T} \epsilon t \qquad \text{(use corollary)} \\
&\leq \max_{\Gamma_T} u_\epsilon + \epsilon T \\
&\leq \max_{\Gamma_T} u + \epsilon T
\end{aligned}$$

Let $\epsilon \to 0$, then $\max_{\overline{U}_T} u \leq \max_{\Gamma_T} u$. Since $\Gamma_T \subset \overline{U}_T$, we conclude $\max_{\overline{U}_T} u = \max_{\Gamma_T} u$

17. **NOTE :** The reader should read the proof of THEOREM 3(A mean-value property for the heat equation) at 2.3.2 before start reading the solution.

(a) We modify the proof of the THEOREM 3. Put v instead of u in the proof. We know that

$$\begin{aligned}
\phi'(r) &= A + B \\
&= \frac{1}{r^{n+1}} \int\int_{E(r)} -4n v_s \psi - \frac{2n}{s} \sum_{i=1}^{n} v_{y_i} y_i dyds
\end{aligned}$$

Since ψ defined to be

$$\psi = -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r = \log(\Phi(y, -s) r^n)$$

$\psi \geq 0$ in $E(r)$ because $\Phi(y, -s) r^n \geq 1$ in $E(r)$. Thus $4n\psi(v_s - \Delta v) \leq 0$, $-4n\psi v_s \geq -4n\Delta v$. Then we have inequality

$$\begin{aligned}
\phi'(r) &= \frac{1}{r^{n+1}} \int\int_{E(r)} -4n v_s \psi - \frac{2n}{s} \sum_{i=1}^{n} v_{y_i} y_i dyds \\
&\geq \frac{1}{r^{n+1}} \int\int_{E(r)} -4n \Delta v \psi - \frac{2n}{s} \sum_{i=1}^{n} v_{y_i} y_i dyds \\
&= 0
\end{aligned}$$

according to the proof of the Theorem 3. So we have

$$\phi(r) \geq \phi(\epsilon)$$

for all $r > \epsilon > 0$. But we know

$$\lim_{\epsilon \to 0} \phi(\epsilon) = 4v(0, 0)$$

So we have inequality

$$\frac{1}{r^n} \int\int_{E(r)} v(y, s) \frac{|y|^2}{s^2} dyds = \phi(r) \geq 4v(0, 0)$$

At first, the book choses $x = t = 0$ without losing generality, so we

$$\frac{1}{4r^n} \int\int_{E(x,t;r)} v(y, s) \frac{|x - y|^2}{(t - s)^2} dyds \geq v(x, t)$$

(b) We modify the proof of the THEOREM 4 at 2.3.2 (Strong Maximum Principle for the Heat Equation). In the proof, put v instead of u. Assume that there exist a point $(x_0, t_0) \in U_T$ with $v(x_0, t_0) = M := \max_{\overline{U_T}} u$. Then for sufficiently small $r > 0$, $E(x_0, t_0; r) \subset U_T$, thus

$$\begin{aligned} M &= v(x_0, t_0) \\ &\leq \frac{1}{4r^n} \int\int_{E(x_0,t_0;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2} dyds \\ &\leq M \frac{1}{4r^n} \int\int_{E(x_0,t_0;r)} \frac{|x-y|^2}{(t-s)^2} dyds \\ &= M \end{aligned}$$

so we must have $v(y, s) = M$ for $(y, s) \in E(x_0, t_0; r)$. The rest of the proof is the same.

(c) Follow the equations

$$v_t = \phi'(u) u_t$$
$$v_{x_i} = \phi'(u) u_{x_i}$$
$$v_{x_i x_i} = \phi''(u) u_{x_i}^2 + \phi'(u) u_{x_i x_i}$$

Thus we have

$$v_t - \Delta v = \phi'(u) u_t - \phi'(u) \Delta u - \phi''(u) \sum_{i=1}^n u_{x_i}^2 \leq 0$$

(d) $v = |Du|^2 + u_t^2 = u_t^2 + \sum_{j=1}^n u_{x_j}^2$, so we

$$v_t = 2u_t u_{tt} + 2 \sum_{j=1}^n u_{x_j} u_{x_j t} \qquad (17.1)$$

$$v_{x_i} = 2u_t u_{t x_i} + 2 \sum_{j=1}^n u_{x_j} u_{x_j x_i}$$

$$v_{x_i x_i} = 2\big(u_t u_{t x_i x_i} + u_{t x_i}^2 + \sum_{j=1}^n u_{x_j} u_{x_j x_i x_i} + u_{x_j x_i}^2\big)$$

so we have

$$\begin{aligned} \Delta v &= \sum_{i=1}^n v_{x_i x_i} \\ &= 2 \sum_{i=1}^n \left(u_t u_{t x_i x_i} + u_{t x_i}^2 + \sum_{j=1}^n u_{x_j} u_{x_j x_i x_i} + u_{x_j x_i}^2 \right) \\ &= 2 \left(\sum_{i=1}^n u_{t x_i}^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 + u_t \Delta u_t + \sum_{j=1}^n u_{x_j} \Delta u_{x_j} \right) \qquad (17.2) \end{aligned}$$

since u solves heat equation, so does u_t and u_{x_i} for $i = 1, 2, ..., n$. This is true since differentiation commutes. Thus (17.2) becomes

$$\begin{aligned} \Delta v &= 2 \left(\sum_{i=1}^n u_{t x_i}^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 + u_t \Delta u_t + \sum_{j=1}^n u_{x_j} \Delta u_{x_j} \right) \\ &= 2 \left(\sum_{i=1}^n u_{t x_i}^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 \right) + 2u_t u_{tt} + 2 \sum_{j=1}^n u_{x_j} u_{x_j t} \\ &\geq 2u_t u_{tt} + 2 \sum_{j=1}^n u_{x_j} u_{x_j t} \\ &= v_t \qquad \text{(by 17.1)} \end{aligned}$$

thus we have $\Delta v \geq v_t$, so v is a subsolution.

18. Since u is smooth, differentiation commutes, so

$$\begin{aligned} v_{tt} - \Delta v &= u_{ttt} - \Delta u_t \\ &= \frac{\partial}{\partial t}(u_{tt} - \Delta u) \\ &= 0 \end{aligned}$$

Moreover, $v = u_t = h$ on $\Re^n \times \{t = 0\}$ and

$$v_t(x,0) = \frac{\partial}{\partial t} u_t(x,0) = \frac{\partial}{\partial t} h(x) = 0$$

so $v_t = 0$ on $\Re^n \times \{t = 0\}$.

19. (a) We have $0 = u_{xy} = \frac{\partial}{\partial x} u_y$, so u_y is constant as x changes, thus u_y only depends on y. Say $u_y(x,y) = c(y)$. By Fundamental Theorem of Calculus, we have

$$u(x,y) - u(x,0) = \int_0^y u_y(x,z)dz = \int_0^y c(z)dz \qquad (19.1)$$

Let $F(x) := u(x,0)$ and $G(y) := \int_0^y c(z)dz$. By 19.1, we have $u(x,y) = F(x) + G(y)$.

(b) ξ and η are defined to be $\xi = \xi(x,t) = x + t$ and $\eta = \eta(x,t) = x - t$. We have an equality

$$u(x,t) = u(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2})$$

Thus we have

$$u_\xi = u_\xi(x,t) = \frac{1}{2} u_x(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}) + \frac{1}{2} u_t(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2})$$

$$u_{\xi\eta} = u_{\xi\eta}(x,t) = \frac{1}{2}(\frac{1}{2} u_{xx} - \frac{1}{2} u_{xt}) + \frac{1}{2}(\frac{1}{2} u_{tx} - \frac{1}{2} u_{tt}) = \frac{1}{4}(u_{xx} - u_{tt})$$

Now it is clear to see

$$u_{tt} - u_{xx} = 0 \iff u_{\xi\eta} = 0$$

SOLUTIONS OF CHAPTER 3

1. Trivially $u_t(x,t,a,b) = -H(a)$ and $Du(x,t,a,b) = a$ so $u_t + H(Du) = -H(a) + H(a) = 0$. Denote $y = (x,t) \in \Re^{n+1}, c = (a,b) \in \Re^{n+1}$ and $u(y,c) = u(x,t,a,b)$. Only thing left to prove is $\text{rank}(D_c u, D^2_{yc} u) = n+1$

$$D_c u, D^2_{yc} u = \begin{bmatrix} x_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ x_2 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n & 0 & 0 & 0 & \cdots & 1 & 0 \\ 1 & -H_{a_1}(a) & -H_{a_2}(a) & -H_{a_3}(a) & \cdots & -H_{a_n}(a) & 0 \end{bmatrix}$$

The matrix $D_c u, D^2_{ya} u$ (i.e. the left $(n+1) \times (n+1)$ part of the matrix $D_c u, D^2_{yc} u$) has determinant $(-1)^n$, so it has rank $n+1$, so we are done.

2. (i) for $u(x;a) = x_1 + a^2 x_2 - 2a$
$D_a u(x;a) = 2ax_2 - 2$ so for $\phi(x) = \phi(x_1, x_2) = \frac{1}{x_2}$, the function solves the equation $D_a u(x;\phi(x)) = 0$. The envelope is

$$v(x) = u(x;\phi(x)) = u(x_1, x_2; \frac{1}{x_2}) = x_1 - \frac{1}{x_2}$$

(ii) for $u(x;a) = 2a_1x_1 + 2a_2x_2 - x_3 + a_1^2 + a_2^2$
$D_a u(x;a) = (2x_1 + 2a_1, 2x_2 + 2a_2)$ so the function $\phi(x) = \phi(x_1,x_2,x_3) = (-x_1, -x_2)$ solves the equation $D_a u(x;\phi(x)) = 0$. The envelope is

$$v(x) = u(x;\phi(x)) = -x_1^2 - x_2^2 + x_3$$

3. (a) Let's write the equation

$$G(x,u,a) = G(x, u(x,a), a) = 0$$

$$G_{x_i}(x,u,a) + G_z(x,u,a) u_{x_i} = 0$$

for the function $G(x,z,a) = \sum_{j=1}^n a_j x_j^2 + z^3$ The equation is

$$2a_i x_i + 3u^2 u_{x_i} = 0$$

Therefore

$$\frac{3}{2} u^2 u_{x_i} x_i = -a_i x_i^2$$

if we add up for all i's, we find that

$$\frac{3}{2} u^2 Du \cdot x = \sum_{i=1}^n \frac{3}{2} u^2 u_{x_i} x_i = \sum_{i=1}^n -a_i x_i^2 = u^3$$

Therefore $u(x,a)$ solves the PDE

$$F(Du,u,x) = \frac{3}{2} Du \cdot x - u = 0$$

(b) to be continued...

4. (a) Let's write $y = (x,t)$ and consider characteristic equations for $y(s) = (x(s), t(s))$, $q(s) = (Du_x(y(s)), u_t(y(s)))$ and $z(s) = u(y(s))$ Then the PDE becomes

$$F(q,z,y) = q \cdot (b,1) - f(y) = 0$$

Therefore

$$D_qF(q,z,y) = (b,1)$$
$$D_yF(q,z,y) = D_yf(y)$$
$$D_zF(q,z,y) = 0$$

Then the characteristic equations are

$$\dot{y}(s) = (b,1)$$
$$z'(s) = \dot{z}(s) = (b,1)\cdot q(s) = b\cdot Du(y(s)) + u_t(y(s)) = f(y(s))$$
$$\dot{q}(s) = -D_yf(y(s))$$

(b) Let's choose $y(s) = (bs+c,s)$ for some constant $c \in \Re^n$. We need to find proper s and s to have equality $y(s) = (x,t)$. Clearly $t = s$ and $c = x - bt$. Therefore

$$y(s) = (x + b(s-t), s)$$

By Fundamental theorem of calculus,

$$\begin{aligned} z(t) - z(0) &= \int_0^t z'(s)ds \\ &= \int_0^t f(z(s))ds \\ &= \int_0^t f(x+b(s-t),s)ds \end{aligned}$$

But $z(0) = u(y(0)) = u(x-bt,0) = g(x-bt)$, so

$$u(x,t) = z(t) = g(x-bt) + \int_0^t f(x+b(s-t),s)ds$$

which agrees with the formula in 2.1.2

5. (a) Let's read the equation as

$$F(Du,u,x) = x\cdot Du - 2u = 0$$

so the function $F(p,z,x)$ satisfies

$$F(p,z,x) = p\cdot x - 2z = 0$$
$$D_pF(p,z,x) = x$$
$$D_xF(p,z,x) = p$$
$$D_zF(p,z,x) = -2$$

Therefore the characteristic equations are

$$\dot{x}(s) = x(s)$$

$$\dot{z}(s) = x(s)\cdot p(s) = 2z(s)$$
$$\dot{p}(s) = -p(s) - (-2)p(s) = p(s)$$

Let's choose

$$x(s) = (x_1(s), x_2(s)) = (C_1e^s, e^s)$$

for some constant C_1. Moreover, $z(s)$ must be

$$z(s) = C_2e^{2s}$$

but

$$g(C_1) = u(C_1, 1) = u(x(0)) = z(0) = C_2$$

so

$$z(s) = g(C_1)e^{2s}$$

For given (a, b), if we choose $s = \log(b)$ and $C_1 = \frac{a}{b}$, we find that

$$x(s) = x(\log(b)) = (C_1 b, b) = (a, b)$$

Thus

$$u(a, b) = u(x(\log(b))) = z(\log(b)) = g(C_1)e^{2\log(b)} = g(\frac{a}{b})b^2$$

Finally, let's prove that the function $u(x_1, x_2) = g(\frac{x_1}{x_2})x_2^2$ indeed solves the equation system

$$x \cdot Du - 2u = 0$$

$$u(x_1, 1) = g(x_1)$$

The second equation is trivial, for the first equation

$$x_1 u_{x_1} + x_2 u_{x_2} - 2u = x_1 g'(\frac{x_1}{x_2})\frac{1}{x_2}x_2^2 + x_2(g(\frac{x_1}{x_2})2x_2 + g'(\frac{x_1}{x_2})\frac{-x_1}{x_2^2}x_2^2) - 2g(\frac{x_1}{x_2})x_2^2 = 0$$

(b) We can read the equation as $F(Du, u, x) = 0$ where

$$F(p, z, x) = p \cdot (x_1, 2x_2, 1) - 3z = 0$$

the derivatives are

$$D_p F(p, z, x) = (x_1, 2x_2, 1)$$

$$D_x F(p, z, x) = (p_1, 2p_2, 0)$$

$$D_z F(p, z, x) = -3$$

so the characteristic equations are

$$\dot{x}(s) = (x_1, 2x_2, 1)$$

$$\dot{z}(s) = D_p F(p, z, x) \cdot p = 3z(s)$$

$$\dot{p}(s) = -(p_1, 2p_2, 0) - (-3)p = (2p_1, p_2, 3p_3)$$

Considering equations, we can choose $x(s) = (C_1 e^s, C_2 e^{2s}, s)$ and $z(s) = C_3 e^{3s}$. Putting $s = 0$

$$C_3 = z(0) = u(x(0)) = u(C_1, C_2, 0) = g(C_1, C_2)$$

Let (x_1, x_2, x_3) be given. Only thing left is to choose proper C_1, C_2 and s such that $x(s) = (x_1, x_2, x_3)$. Clearly $s = x_3$, $C_1 = x_1 e^{-x_3}$ and $C_2 = x_2 e^{-2x_3}$. Finally

$$u(x_1, x_2, x_3) = u(x(x_3)) = g(C_1, C_2)e^{3x_3} = g(x_1 e^{-x_3}, x_2 e^{-2x_3})e^{3x_3}$$

Let's prove u indeed solves the PDE.

$$x_1 u_{x_1} = x_1 g_{x_1}(x_1 e^{-x_3}, x_2 e^{-2x_3})e^{2x_3}$$

$$2x_2 u_{x_2} = 2x_2 g_{x_2}(x_1 e^{-x_3}, x_2 e^{-2x_3})e^{x_3}$$

$$u_{x_3} = -g_{x_1} x_1 e^{-x_1} e^{3x_3} - g_{x_2} 2x_2 e^{-2x_3})e^{3x_3} + 3ge^{3x_3} = -x_1 g_{x_1} e^{2x_2} - 2x_2 e^{x_3} + 3ge^{3x_3}$$

Therefore $F(Du, u, x) = x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} - 3u = 0$

(c) We can read the equation as $F(Du, u, x) = 0$ where

$$F(p, z, x) = p \cdot (z, 1) - 1$$

the derivatives are

$$D_pF(p,z,x) = (z,1)$$
$$D_xF(p,z,x) = 0$$
$$D_zF(p,z,x) = p_1$$

thus the characteristic equations are

$$\dot{x}(s) = (z(s),1)$$
$$\dot{z}(s) = 1$$

Choose $z(s) = s + c_1$ and $x_2(s) = s$. Since $x_1'(s) = z(s) = s + c_1$, choose $x_1(s) = \frac{1}{2}(x+c_1)^2 + c_2$. Assume that for some s, $x_1(s) = s = x_2(s)$, then

$$s + c_1 = z(s) = u(x(s)) = u(s,s) = \frac{1}{2}s$$

so s must be $-2c_1$. Let's put $s = -2c_1$ in the equation $x_1(s) = s$ to find c_2.

$$-2c_1 = x(-2c_1) = \frac{1}{2}c_1^2 + c_2$$

. Thus $c_2 = -2c_1 - \frac{1}{2}c_1^2$. Let's write $c = c_1$ to simplify the notation. In summation, we find that

$$x(s) = (\frac{1}{2}(s+c)^2 - \frac{1}{2}c^2 - 2c, s) = (\frac{1}{2}s^2 + sc - 2c, s)$$

and

$$u(\frac{1}{2}s^2 + sc - 2c, s) = u(x(s)) = z(s) = s + c$$

We need to find proper s and c for given (x_1, x_2) such that $x(s) = (x_1, x_2)$.

$$(x_1, x_2) = x(s) = (\frac{1}{2}s^2 + sc - 2c, s)$$

so $x_2 = s$ and $c(s-2) = x_1 - \frac{1}{2}s^2 = x_1 - \frac{1}{2}x_2^2$. Finally, for $x_2 \neq 2$,

$$c = \frac{2x_1 - x_2^2}{2(x_2 - 2)}$$

and

$$u(x_1, x_2) = u(x(x_2))z(x_2) = x_2 + c = x_2 + \frac{2x_1 - x_2^2}{2(x_2-2)} = \frac{x_2^2 - 4x_2 + 2x_1}{2(x_2 - 2)}$$

Let's show that $u = \frac{x_2^2 - 4x_2 + 2x_1}{2(x_2-2)}$ is indeed a solution. For $x_1 = x_2 \neq 2$, clearly $u(x_1, x_1) = \frac{x_1}{2}$. Also

$$u_{x_1}(x_1, x_2) = \frac{1}{x_2 - 2}$$

$$u_{x_2}(x_1, x_2) = \frac{(2x_2 - 4)2(x_2 - 2) - 2(x_2^2 - 4x_2 + 2x_1)}{(2(x_2-2))^2} = 1 - uu_{x_1}$$

Thus $uu_{x_1} + u_{x_2} = 1$, so we are done.

6. **(a)** Let $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, ..., \mathbf{x}^n)$ and $\mathbf{b} = (\mathbf{b}^1, \mathbf{b}^2, ..., \mathbf{b}^n)$

$$J(s,x,t) = \det D_x\mathbf{x}(s,x,t) = \det(\frac{\partial \mathbf{x}^i}{\partial x_j}) = \sum_{\sigma \in S_n} (-1)^{sgn(n)} \prod_{j=1}^{n} \mathbf{x}^j_{x_{\sigma(j)}}$$

therefore J is a linear combination of functions $f_\sigma(s,x,t) = (-1)^{sgn(\sigma)} \prod_{j=1}^n \mathbf{x}^j_{x_{\sigma(j)}}(s,x,t)$ where $\sigma \in S_n$. We need to prove that

$$J_s = \sum_{\sigma \in S_n} \frac{\partial f_\sigma}{\partial s} = \mathrm{div}(\mathbf{b}(\mathbf{x})) \sum_{\sigma \in S_n} f_\sigma$$

Let's start to compute RHS

$$\frac{\partial f_\sigma}{\partial s} = (-1)^{sgn(\sigma)} \frac{\partial}{\partial s} \prod_{j=1}^n \mathbf{x}^j_{x_{\sigma(j)}} = (-1)^{sgn(\sigma)} \sum_{i=1}^n \frac{\partial}{\partial s} \mathbf{x}^i_{x_{\sigma(i)}} \prod_{j=1,\, j\neq i}^n \mathbf{x}^j_{x_{\sigma(j)}}$$

Since $\mathbf{x}$ is smooth, we know that differentiation commutes, so

$$\frac{\partial}{\partial s} \mathbf{x}^i_{x_{\sigma(i)}} = \frac{\partial}{\partial x_{\sigma(i)}} \mathbf{x}^i_s$$

By the given condition $\frac{\partial \mathbf{x}}{\partial s} = \dot{\mathbf{x}} = \mathbf{b}(\mathbf{x})$ we know that $\mathbf{x}^i_s = \mathbf{b}^i(\mathbf{x})$ for $i = 1, 2, ..., n$. Thus

$$\frac{\partial}{\partial s} \mathbf{x}^i_{x_{\sigma(i)}} = \frac{\partial}{\partial x_{\sigma(i)}} \mathbf{x}^i_s = \frac{\partial}{\partial x_{\sigma(i)}} \mathbf{b}^i(\mathbf{x}) = \sum_{k=1}^n \mathbf{b}^i_{x_k} \mathbf{x}^k_{x_{\sigma(i)}}$$

Therefore we find that

$$\frac{\partial f_\sigma}{\partial s} = (-1)^{sgn(\sigma)} \sum_{i=1}^n \left(\left(\sum_{k=1}^n \mathbf{b}^i_{x_k} \mathbf{x}^k_{x_{\sigma(i)}} \right) \prod_{j=1,\, j\neq i}^n \mathbf{x}^j_{x_{\sigma(j)}} \right)$$

Thus

$$\begin{aligned} J_s &= \sum_{\sigma \in S_n} \frac{\partial f_\sigma}{\partial s} \\ &= \sum_{\sigma \in S_n} \left((-1)^{sgn(\sigma)} \sum_{i=1}^n \left(\left(\sum_{k=1}^n \mathbf{b}^i_{x_k} \mathbf{x}^k_{x_{\sigma(i)}} \right) \prod_{j=1,\, j\neq i}^n \mathbf{x}^j_{x_{\sigma(j)}} \right) \right) \\ &= \sum_{i=1}^n \sum_{k=1}^n \mathbf{b}^i_{x_k} \left(\sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \mathbf{x}^k_{x_{\sigma(i)}} \prod_{j=1,\, j\neq i}^n \mathbf{x}^j_{x_{\sigma(j)}} \right) \end{aligned}$$

but

$$\sum_{\sigma \in S_n} (-1)^{sgn(\sigma)} \mathbf{x}^k_{x_{\sigma(i)}} \prod_{j=1,\, j\neq i}^n \mathbf{x}^j_{x_{\sigma(j)}} = \det D_x(\mathbf{x}^1, \mathbf{x}^, ..., \mathbf{x}^{i-1}, \mathbf{x}^k, \mathbf{x}^{i+1}, ..., \mathbf{x}^n)$$

which is equal to J for $i = k$ and 0 for $i \neq k$. Therefore

$$J_s = \sum_{i=1}^n b^i_{x_i} J = \mathrm{div}(\mathbf{b}(\mathbf{x}))J$$

(b) Note that the problem is written wrong, errata! We'll use characteristic equations.

$$u_t + \mathrm{div}(u\mathbf{b}) = u_t + Du \cdot \mathbf{b} + u\mathrm{div}(\mathbf{b})$$

We can read the equation as

$$F(Du, u_t, u, x, t) = 0$$

where for $q, y \in \Re^{n+1}$ and $z \in \Re$

$$F(q, z, y) = q \cdot (\mathbf{b}, 1) + z\mathrm{div}(\mathbf{b}) = 0$$

The derivatives are

$$D_q F(q, z, y) = (\mathbf{b}, 1)$$

$$D_z F(q,z,y) = \text{div}(\mathbf{b})$$
$$D_y F(q,z,y) = 0$$

Thus the characteristic equations are

$$\dot{y}(s) = (\mathbf{b}, 1)$$
$$z_s(s) = \dot{z}(s) = (\mathbf{b}, 1)\cdot q = -z(s)\text{div}(\mathbf{b})$$

Since we are given the fact that $\dot{\mathbf{x}} = \mathbf{b}(\mathbf{x})$, we can choose $y(s) = (\mathbf{x}(s), s)$

claim: $z(s)J(s)$ is a constant function.

proof: We use Euler Formula, which we proved in **(a)**

$$\frac{\partial}{\partial s}(zJ) = z_s J + zJ_s = -z\text{div}(\mathbf{b})J + z\text{div}(\mathbf{b})J = 0$$

For given $(x,t) \in \Re^n \times [0,\infty)$, choose $\mathbf{x}(s) = \mathbf{x}(s,x,t)$. Since zJ is constant

$$z(t)J(t) = z(0)J(0)$$

but

$$z(t) = u(\mathbf{x}(t),t) = u(\mathbf{x}(t,x,t),t) = u(x,t)$$
$$J(t,x,t) = \det D_x \mathbf{x}(t,x,t) = \det D_x(x) = 1$$

so

$$\begin{aligned} u(x,t) &= z(t)J(t) \\ &= z(0)J(0) \\ &= u(\mathbf{x}(0,x,t),0)J(0,x,t) \\ &= g(\mathbf{x}(0,x,t))J(0,x,t) \end{aligned}$$

since $u = g$ on $\Re^n \times \{t = 0\}$

7. NOTE: Before start reading the solution, the reader should read appendix C.1, and section 3.2.3, especially Lemma 1.

Define functions Φ and Ψ as in appendix C.1. Define $v(y) = u(\Psi(y)) : V \to \Re$ as in 3.1.3.a. Then by (26), we have an equality

$$F(Dv(y)D\Phi(\Psi(y)), v(y), \Psi(y)) = F(Du(\Psi(y)), u(\Psi(y)), \Psi(y)) = 0 \qquad (7.1)$$

Define function $G : \Re^{2n+1} \to \Re$ such that

$$G(q,z,y) = F(qD\Phi(\Psi(y)), z, \Psi(y)) \qquad (7.2)$$

By 7.1, we have

$$G(Dv(y), v(y), y) = 0$$

Moreover $V := \Phi(U)$ is flat near $x^0 \in \Gamma$. Thus by LEMMA 1, we have noncharacteristic boundary condition

$$G_{q_n}(q^0, z^0, y^0) \neq 0$$

where $y^0 = \Phi(x^0)$ and $q^0 = p^0 D\Psi(y^0)$. By the definition of Φ

$$D\Phi(x) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -\gamma_{x_1} & -\gamma_{x_2} & -\gamma_{x_3} & \cdots & -\gamma_{x_{n-1}} & 1 \end{bmatrix}$$

Let $c_i(x)$ define the i^{th} column of the matrix. Then we have

$$\frac{\partial}{\partial q_n}(q \cdot c_j(x)^T) = \begin{cases} -\gamma_{x_j}(x_1, x_2, ..., x_{n-1}) & \text{for } j = 1, 2, ..., n \\ 1 & \text{for } j = n \end{cases}$$

By equality in 7.2

$$\begin{aligned} G_{q_n}(q, z, y) &= \frac{\partial}{\partial q_n} F(qD\Phi(\Psi(y)), z, \Psi(y)) \\ &= \sum_{j=1}^{n} F_{q_j}(qD\Phi(\Psi(y)), z, \Psi(y)) \frac{\partial}{\partial q_n}(q \cdot c_i(\Psi(y)^T) \end{aligned} \tag{7.3}$$

In particular, when x is near x^0, the equation $x_n - \gamma(x_1, x_2, ..., x_{n-1}) = 0$ defines the boundary for U. Thus $\mathbf{v}(x^0) = (-\gamma_{x_1}, -\gamma_{x_2}, ..., -\gamma_{x_{n-1}}, 1)$ is the normal vector at the point x^0. Using 7.3

$$G_{q_n}(q^0, z^0, y^0) = \sum_{j=1}^{n} F_{q_j}(q^0 D\Phi(\Psi(y^0)), z^0, \Psi(y^0)) \frac{\partial}{\partial q_j}(q^0 D\Phi(\Psi(y^0)))$$

But by definitions of y^0 and q^0

$$\Psi(y^0) = \Psi(\Phi(x^0))x^0$$

$$q^0 D\Phi(\Psi(y^0)) = p^0 D\Psi(x^0) D\Phi(x^0) = p^0$$

Thus

$$G_{q_n}(q^0, z^0, y^0) = F_p(p^0, z^0, x^0) \cdot \mathbf{v}(x^0)$$

Therefore the noncharacteristic condition becomes

$$F_p(p^0, z^0, x^0) \cdot \mathbf{v}(x^0) \neq 0$$

when Γ is not flat near x^0.

8. (Note that there is an Errata in the statement of problem). We shall prove that the function u which satisfies the condition

$$u = u(x,t) = g(x - t\mathbf{F}'(u)) = g(x - t\mathbf{F}'(u(x,t)))$$

provides implicit solution of the conservation law

$$u_t + \text{div}\mathbf{F}(u) = u_t + Du \cdot \mathbf{F}' = 0$$

Let's calculate $u_t(x,t)$

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t} g(x - t\mathbf{F}'(u)) \\ &= \sum_{i=1}^{n} g_{x_i}(x - t\mathbf{F}'(u)) \frac{\partial}{\partial t}(x_i - t(\mathbf{F}^i)'(u)) \\ &= \sum_{i=1}^{n} g_{x_i}(x - t\mathbf{F}'(u))(-(\mathbf{F}^i)'(u) - t(\mathbf{F}^i)''(u)u_t) \\ &= -Dg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}'(u) - tu_t Dg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}''(u) \end{aligned}$$

Therefore

$$u_t(x,t)\left(1 + tDg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}''(u)\right) = -Dg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}'(u)$$

Let's define $\gamma(x,t) = 1 + tDg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}''(u)$ to make equation looks simpler. Basically,

$$u_t(x,t)\gamma(x,t) = -Dg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}'(u) \qquad (8.1)$$

Let's calculate $\text{div}\mathbf{F}(u) = Du \cdot \mathbf{F}'(u)$. For $i = 1, 2, ..., n$

$$\begin{aligned}
u_{x_i}(x,t) &= \frac{\partial}{\partial x_i} g(x - t\mathbf{F}'(u)) \\
&= \sum_{j=1}^{n} g_{x_j}(x - t\mathbf{F}'(u)) \frac{\partial}{\partial x_i}(x_j - t(\mathbf{F}^j)'(u)) \\
&= g_{x_i}(x - t\mathbf{F}'(u))(1 - t(\mathbf{F}^i)''(u)u_{x_i}) + \sum_{j=1, j\neq i}^{n} g_{x_j}(x - t\mathbf{F}'(u))(-t(\mathbf{F}^j)''(u)u_{x_i}) \\
&= g_{x_i}(x - t\mathbf{F}'(u)) - tu_{x_i} \sum_{j=1}^{n} g_{x_j}(x - t\mathbf{F}'(u))(\mathbf{F}^j)''(u) \\
&= g_{x_i}(x - t\mathbf{F}'(u)) - tu_{x_i} Du(x - t\mathbf{F}'(u))\mathbf{F}''(u)
\end{aligned}$$

Therefore

$$\gamma(x,t)u_{x_i}(x,t) = g_{x_i}(x - t\mathbf{F}'(u))$$

Thus

$$\begin{aligned}
\gamma(x,t)\text{div}\mathbf{F}(u) &= \gamma(x,t)Du \cdot \mathbf{F}'(u) \\
&= \sum_{i=1}^{n} \gamma(x,t)u_{x_i}(x,t)(\mathbf{F}^i)'(u) \\
&= \sum_{i=1}^{n} g_{x_i}(x - t\mathbf{F}'(u))(\mathbf{F}^i)'(u) \\
&= Dg(x - t\mathbf{F}'(u))\mathbf{F}'(u)
\end{aligned}$$

Using (8.1), we conclude that $\gamma(x,t)\Big(u_t + \text{div}\mathbf{F}(u)\Big) = 0$. If we are given that

$$\gamma(x,t) = 1 + tDg(x - t\mathbf{F}'(u)) \cdot \mathbf{F}''(u) \neq 0$$

we can say that $u_t + \text{div}\mathbf{F}(u) = 0$, i.e. the function u solves the conservation law.

9. NOTE: Before start reading the solution, the reader should know the proof of the THEOREM 1 (Euler-Lagrange Equations) in section 3.3.1.

(a) Let's define $\mathcal{S} := \{\mathbf{y} \in C^\infty([0,t];\Re^n) \big| \mathbf{y}(0) = \mathbf{y}(t) = 0\}$ and $i : \Re \to \Re$ such that

$$i(\tau) := I[\mathbf{x}(\cdot) + \tau\mathbf{y}(\cdot)] = \int_0^t L(\dot{\mathbf{x}}(s) + \tau\dot{\mathbf{y}}(s), \mathbf{x}(s) + \tau\mathbf{y}(s))ds$$

Clearly for any $\mathbf{y} \in \mathcal{S}$ and $\tau \in \Re$, $\mathbf{x} + \tau\mathbf{y} \in \mathcal{A}$. Therefore $i(\tau)$ has minimum at $\tau = 0$ for any fixed $\mathbf{y} \in \mathcal{P}$, so we know $i'(\tau) = 0$. As we see in the proof of the THEOREM 1

$$i'(\tau) = \int_0^t \sum_{i=1}^{n} L_{v_i}(\dot{\mathbf{x}} + \tau\dot{\mathbf{y}}, \mathbf{x} + \tau\mathbf{y})\dot{\mathbf{y}}^i + L_{x_i}(\dot{\mathbf{x}} + \tau\dot{\mathbf{y}}, \mathbf{x} + \tau\mathbf{y})\mathbf{y}^i$$

Thus at $\tau = 0$

$$0 = i(0) = \int_0^t \sum_{i=1}^{n} L_{v_i}(\dot{\mathbf{x}}, \mathbf{x})\dot{\mathbf{y}}^i + L_{x_i}(\dot{\mathbf{x}}, \mathbf{x})\mathbf{y}^i ds \qquad (9.1)$$

By integration by parts

$$\begin{aligned} 0 &= L_{v_i}(\dot{\mathbf{x}}(t), \mathbf{x}(t))\mathbf{y}^i(t) - L_{v_i}(\dot{\mathbf{x}}(0), \mathbf{x}(0))\mathbf{y}^i(0) \\ &= \int_0^t \left[\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s))\mathbf{y}^i(s) \right] ds \\ &= \int_0^t \left[\left(\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)\right) \mathbf{y}^i(s) + L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s))\dot{\mathbf{y}}^i(s) \right] ds \end{aligned}$$

Therefore

$$\int_0^t L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s))\dot{\mathbf{y}}^i(s) ds = -\int_0^t \left(\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \right) \mathbf{y}^i(s) ds$$

If we put this equality in (9.1), we find that

$$0 = \sum_{i=1}^n \int_0^t \left[-\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) + L_{x_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \right] \mathbf{y}^i ds$$

Since this equality is true for any smooth function $\mathbf{y} \in \mathcal{S}$, we conclude that

$$-\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) + L_{x_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) = 0$$

for all $i = 1, 2, .., n$. In other words, $\mathbf{x}(s)$ satisfies Euler-Lagrange Equations.

(b) Now consider the set $\mathcal{P} := \{\mathbf{y} \in C^\infty([0,t]; \Re^n) \big| \mathbf{y}(t) = 0\}$. Clearly for any $\mathbf{y} \in \mathcal{P}$ and $\tau \in \Re$, $\mathbf{x} + \tau\mathbf{y} \in \mathcal{A}$, so $i(\tau)$ has minimum at $\tau = 0$ for any fixed $\mathbf{y} \in \mathcal{P}$. Thus $i'(0) = 0$. By integration by parts

$$-L_{v_i}(\dot{\mathbf{x}}(0), \mathbf{x}(0))\mathbf{y}^i(0) = \int_0^t \left[\left(\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)\right) \mathbf{y}^i(s) + L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s))\dot{\mathbf{y}}^i(s) \right] ds$$

Therefore

$$\int_0^t L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s))\dot{\mathbf{y}}^i(s) ds = -\int_0^t \left(\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \right) \mathbf{y}^i(s) ds - L_{v_i}(\dot{\mathbf{x}}(0), \mathbf{x}(0))\mathbf{y}^i(0)$$

If we put this equality in (9.1), we find that

$$\begin{aligned} 0 &= i'(0) \\ &= \sum_{i=1}^n \int_0^t \left[-\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) + L_{x_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \right] \mathbf{y}^i ds - D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \cdot \mathbf{y}(0) \end{aligned}$$

But $-\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) + L_{x_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) = 0$ by **(a)**, so $D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \cdot \mathbf{y}(0) = 0$ for any $\mathbf{y} \in \mathcal{P}$, thus $D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) = 0$

(c) For any $\mathbf{y} \in \mathcal{S}$ and $\tau \in \Re$, $\mathbf{x} + \tau\mathbf{y} \in \mathcal{A}$ and $g(\mathbf{x}(0)) = g(\mathbf{x}(0) + \tau\mathbf{y}(0))$. Define $j : \Re \to \Re$ such that

$$j(\tau) = i(\tau) + g(\mathbf{x}(0) + \tau\mathbf{y}(0)) = \int_0^t L(\dot{\mathbf{x}}(s) + \tau\dot{\mathbf{y}}(s), \mathbf{x}(s) + \tau\mathbf{y}(s)) ds + g(\mathbf{x}(0) + \tau\mathbf{y}(0))$$

By similar reasoning as in **(a)**,

$$0 = j'(0) = \sum_{i=1}^n \int_0^t \left[-\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) + L_{x_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \right] \mathbf{y}^i ds$$

notice that we got rid of $g(\mathbf{x}(0) + \tau\mathbf{y}(0))$ since it is constant as τ changes. Therefore $\mathbf{x}$ satisfies Euler-Lagrange equations.

Now consider $\mathbf{y} \in \mathcal{P}$ and define $j(\tau)$ in the same way. Then

$$j'(\tau) = i'(\tau) + \frac{\partial}{\partial\tau} g(\mathbf{x}(0) + \tau\mathbf{y}(0)) = i'(\tau) + Dg(\mathbf{x}(0) + \tau\mathbf{y}(0)) \cdot \mathbf{y}(0)$$

So at $\tau = 0$, $j'(0) = i'(0) + Dg(\mathbf{x}(0)) \cdot \mathbf{y}(0)$. But from **(b)**, we know

$$\begin{aligned} i'(0) &= \sum_{i=1}^{n} \int_0^t \Big[-\frac{\partial}{\partial s} L_{v_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) + L_{x_i}(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \Big] \mathbf{y}^i ds - D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \cdot \mathbf{y}(0) \\ &= -D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \cdot \mathbf{y}(0) \end{aligned}$$

since $\mathbf{x}$ satisfies Euler-Lagrange Equations. Therefore

$$\begin{aligned} 0 &= j'(0) \\ &= -D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \cdot \mathbf{y}(0) + Dg(\mathbf{x}(0)) \cdot \mathbf{y}(0) \\ &= \Big(Dg(\mathbf{x}(0)) - D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) \Big) \cdot \mathbf{y}(0) \end{aligned}$$

Since the equality is true for any $\mathbf{y} \in \mathcal{P}$, we must have $D_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) = Dg(\mathbf{x}(0))$ as an initial condition for minimizer $\mathbf{x}$.

10. **(a)** Firstly, let's remember $Young's\, Inequality$: Let $p, q \in (1, \infty)$ and $a, b \in [0, \infty)$, then the inequality

$$\frac{a^p}{p} + \frac{b^p}{p} \geq ab$$

holds. By the definition,

$$L(v) = \sup_{p \in \Re^n} \{v \cdot p - H(p)\} = \sup_{p \in \Re^n} \{v \cdot p - \frac{|p|^r}{r}\}$$

By $Young's\, Inequality$ and $Cauchy's\, Inequality$, we know that $\frac{|p|^r}{r} + \frac{|v|^s}{s} \geq |p||v| \geq p \cdot v$, so

$$L(v) = \sup_{p \in \Re^n} \{v \cdot p - \frac{|p|^r}{r}\} \leq \frac{|v|^s}{s} \qquad (10.1)$$

. If we put $p = v|v|^{\frac{s-r}{r}}$

$$\begin{aligned} v \cdot p - \frac{|p|^r}{r} &= v \cdot v|v|^{\frac{s-r}{r}} - \frac{\left| v|v|^{\frac{s-r}{r}} \right|^r}{r} \\ &= |v|^{\frac{s+r}{r}} - \frac{|v|^s}{r} \qquad (10.2) \\ &= |v|^s - \frac{|v|^s}{r} \\ &= \frac{|v|^s}{s} \end{aligned}$$

At (10.2), we used $\frac{s+r}{r} = \frac{sr}{r} = s$. Thus $L(v) \geq \frac{|v|^s}{s}$. Combining (10.1), we conclude that $L(v) = \frac{|v|^s}{s}$.

(b) We need to determine

$$L(v) := H^*(v) = \sup_{p \in \Re^n} \{v \cdot p - H(p)\}$$

(For the sake of the easiness of the notation, we assume that v and p are column vectors) Fix v and define function $f = f_v : \Re^n \to \Re$ such that

$$f(p) = v \cdot p - H(p) = \sum_{i=1}^{n} (v_i - b_i)p_i - \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} p_i p_j$$

Notice two things: $f \in C^\infty(\Re^n; R)$, and $\sum_{i,j=1}^{n} a_{ij} p_i p_j = p^T A p$. Assume that we also know f is bounded. Then f must have maximum, i.e there exist p^* such that $f(p^*) = \sup_{p \in \Re^n} f(p) = L(v)$. Since f is smooth, p^* must be a critical point, so we have $Df(p^*) = 0$. Let c_i denote i^{th} column of the matrix A (and also i^{th} row since A is a symmetric matrix). Then by simple calculation

$$\frac{\partial}{\partial p_i} f(p^*) = (v_i - b_i) - r_i \cdot p^*$$

Thus

$$Df(p^*) = v - b - Ap^* \qquad (10.3)$$

Since A is a positive definite matrix, it is also invertible. So we can solve the equation (10.3) for $p^* = A^{-1}(v - b)$. Moreover, for this particular choice of p^*, $Hess(f)(p^*) = -A$, which is negative definite, so p is actually local maximum. Since it is only critical point and we assumed f to be bounded, p^* must be the global maximum. Therefore

$$\begin{aligned} L(v) &= f(p^*) \\ &= (v - b) \cdot p^* - \frac{1}{2}(p^*)^T A p^* \\ &= (v - b) \cdot (A^{-1}(v - b)) - \frac{1}{2}(A^{-1}(v - b))^T A(A^{-1}(v - b)) \qquad (10.4) \\ &= (v - b)^T (A^{-1}(v - b)) - \frac{1}{2}((v - b)^T A^{-1})A(A^{-1}(v - b)) \\ &= \frac{1}{2}(v - b)^T A^{-1}(v - b) \end{aligned}$$

At (10.4), we used that

$$(A^{-1}(v - b)) = (v - b)^T (A^{-1})^T$$

and

$$\begin{aligned} (A^{-1})^T &= A^{-1} A (A^{-1})^T \\ &= A^{-1} A^T (A^{-1})^T \\ &= A^{-1} (A^{-1} A)^T \\ &= A^{-1} (I)^T \\ &= A^{-1} \end{aligned}$$

(briefly we proved that if A is symmetric, so is A^{-1}) We only need to prove that f is bounded. Since the function $x \to x^T A x$ is continuous and the set $B(0, 1) \subset \Re^n$ is compact, $x^T A x$ is bounded on the set $B(0, 1)$. Assume $x^T A x > M$ for $|x| = 1$. Since A is positive definite, we can choose $M > 0$. Let $p_1 = \frac{p}{|p|}$

$$\begin{aligned} f(p) &= (v - b) \cdot p - p^T A p \\ &\leq |v - b||p| - |p|^2 p_1^T A p_1 \qquad \text{(by Cauchy's Inequality)} \\ &< |v - b||p| - M|p|^2 \\ &= \frac{1}{M} M|p|(|v - b| - M|p|) \\ &\leq \frac{1}{M} \frac{|v - b|^2}{4} \qquad \text{(by Aritmetic-Geometric Mean Inequality)} \end{aligned}$$

Thus f is bounded, so we are done.

11. Assume that $v \in \partial H(p)$. Let's prove $p \cdot v = H(p) + L(v)$. For any $r \in \Re^n$

$$r \cdot v - H(r) \leq r \cdot v - (H(p) + v \cdot (r - p)) = v \cdot p - H(p)$$

Thus

$$L(v) = \sup_{r \in \Re^n} \{v \cdot r - H(r)\} \leq v \cdot p - H(p)$$

but for $r = p$, $v \cdot r - H(r) = v \cdot p - H(p)$, so $L(v) = v \cdot p - H(p)$.

Now assume that $p \cdot v = H(p) + L(v)$, we'll prove $v \in \partial H(p)$. For any $r \in \Re^n$

$$\begin{aligned} v \cdot p - H(p) &= L(v) \\ &= \sup_{s \in \Re^n} \{v \cdot s - H(s)\} \\ &\geq v \cdot r - H(r) \end{aligned}$$

So

$$H(r) \geq H(p) + v \dot{(} r - p)$$

thus $v \in \partial H(p)$. Since we have duality between L and H

$$v \in \partial H(p) \iff p \cdot v = H(p) + L(v) \iff p \in \partial H(v)$$

12. First observe that

$$\max_{p \in \Re^n} \{-H_1(p) - H_2(-p)\} = -\min_{p \in \Re^n} \{H_1(p) + H_2(-p)\}$$

so we need to prove

$$\min_{v \in \Re^n} \{L_1(v) + L_2(v)\} + \min_{p \in \Re^n} \{H_1(p) + H_2(-p)\} = 0$$

By definition of $H_1 = L_1^*$ and $H_2 = L_2^*$, for all $p \in \Re^n$

$$\begin{aligned} H_1(p) + H_2(-p) &= \sup_{v_1 \in \Re^n} \{v_1 \cdot p - L_1(v_1)\} + \sup_{v_2 \in \Re^n} \{-v_2 \cdot p - L_2(v_2)\} \\ &\geq \sup_{v \in \Re^n} \{-L_1(v) - L_2(v)\} \\ &= -\min_{v \in \Re^n} \{L_1(v) + L_2(v)\} \end{aligned}$$

Therefore we have

$$\min_{p \in \Re^n} \{H_1(p) + H_2(-p)\} \geq -\min_{v \in \Re^n} \{L_1(v) + L_2(v)\}$$

so we proved

$$\min_{v \in \Re^n} \{L_1(v) + L_2(v)\} + \min_{p \in \Re^n} \{H_1(p) + H_2(-p)\} \geq 0$$

Next we prove that it is also ≤ 0. First let's remember some facts about Hamiltonian and Lagrangian.

Lemma (THEOREM 3 at 3.3.2) The three statements

$$p \cdot v = L(v) + H(p)$$

$$p = DL(v)$$

$$v = DL(p)$$

are equivalent, provided that $H = L^*$, L is differentiable at v and H is differentiable at p.

Assume that $L_1(v) + L_2(v)$ obtains its min at v^*. In other words

$$\min_{v\in\Re^n}\{L_1(v)+L_2(v)\} = L_1(v^*)+L_2(v^*)$$

Then we must have $0 = D(L_1+L_2)(v^*) = DL_1(v^*) + DL_2(v^*) = 0$. Choose

$$p^* = DL_1(v^*) = -DL_2(v^*)$$

then by the Lemma,

$$L_1(v^*) + H_1(p^*) = v^* \cdot p^*$$
$$L_2(v^*) + H_2(-p^*) = -v^* \cdot p^*$$

so

$$\begin{aligned}
\min_{v\in\Re^n}\{L_1(v)+L_2(v)\} + \min_{p\in\Re^n}\{H_1(p)+H_2(-p)\} &\leq L_1(v^*)+L_2(v^*)+H_1(p^*)+H_2(-p^*) \\
&= v^*\cdot p^* - v^*\cdot p^* \\
&= 0
\end{aligned}$$

thus we are done.

13. Define function $f(y) = tL(\frac{x-y}{t}) + g(y)$ and assume f takes its minimum at y^0. Then we must have $Df(y^0) = 0$. But we know $Df(y) = tDL(\frac{x-y}{t})\frac{-1}{t} + Dg(y) = Dg(y) - DL(\frac{x-y}{t})$, so we have $DL(\frac{x-y^0}{t}) = Dg(y^0)$. Then by the Lemma from the problem **12**

$$DH(Dg(y^0)) = \frac{x-y^0}{t}$$

but we are given

$$R \geq |DH(Dg(y^0))| = \frac{|x-y^0|}{t}$$

which means $y^0 \in B(x, Rt)$. Thus we must have

$$\min_{y\in\Re^n}\{tL(\frac{x-y}{t}) + g(y)\} = \min_{y\in B(x,Rt)}\{tL(\frac{x-y}{t}) + g(y)\}$$

so we are done.

14. We have Hamiltonian $H(p) = |p|^2$. Let's determine $L := H^*$.

Claim $L(v) = \frac{|v|^2}{4}$ for all $v \in \Re^n$

Proof By Arithmetic Mean-Geometric mean and Cauchy inequalities, we have

$$v\cdot p - H(p) = v\cdot p - |p|^2 \leq |v||p| - |p|^2 \leq (\frac{|v|^2}{4} + |p|^2) - |p|^2 = \frac{|v|^2}{4}$$

for all $p \in \Re^n$. Thus we have $L(v) \leq \frac{|v|^2}{4}$. But for $p^* = \frac{v}{2}$, $p^*\cdot v - |p^*|^2 = \frac{|v|^2}{4}$, so we have $L(v) = \frac{|v|^2}{4}$. Let's apply Hopf-Lax formula for u.

$$\begin{aligned}
u(x,t) &= \min_{y\in\Re^n}\{tL(\frac{x-y}{t}) + g(y)\} \\
&= \min_{y\in\Re^n}\{\frac{|x-y|^2}{4t} + g(y)\}
\end{aligned}$$

Since $g(y) = u(y, 0)$ is given in the problem as

$$g(y) = \begin{cases} 0 & \text{if } y \in E \\ \infty & \text{if } y \notin E \end{cases}$$

$$\begin{aligned} u(x,t) &= \min_{y \in \Re^n} \{ \frac{|x-y|^2}{4t} + g(y) \} \\ &= \min_{y \in E} \{ \frac{|x-y|^2}{4t} \} \\ &= \frac{1}{4t} \text{dist}(x,E)^2 \end{aligned}$$

15. We shall fill the gaps in the **Lemma 4** from 3.3.3. Let's prove (36). Define function $H'(p) = H(p) - \frac{\theta}{2}|p|^2$, then for all $\xi \in \Re^n$

$$\xi^T \mathbf{Hess}(H')(p)\xi = \sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i \xi_j - \theta|\xi|^2 \geq 0$$

thus H' is a convex function. So we have

$$\begin{aligned} H(\frac{p_1+p_2}{2}) &= H'(\frac{p_1+p_2}{2}) + \frac{\theta}{8}|p_1+p_2|^2 \\ &\leq \frac{1}{2}H'(p_1) + \frac{1}{2}H'(p_2) + \frac{\theta}{8}|p_1+p_2|^2 \qquad \text{(since } H' \text{ is convex)} \\ &= \frac{1}{2}H(p_1) - \frac{\theta}{4}|p_1|^2 + \frac{1}{2}H(p_2) - \frac{\theta}{4}|p_2|^2 + \frac{\theta}{8}|p_1+p_2|^2 \\ &= \frac{1}{2}H(p_1) + \frac{1}{2}H(p_2) - \frac{\theta}{8}(2|p_1|^2 + 2|p_2|^2 - |p_1+p_2|^2) \\ &= \frac{1}{2}H(p_1) + \frac{1}{2}H(p_2) - \frac{\theta}{8}|p_1-p_2|^2 \end{aligned}$$

so we proved (36). Let's prove (37). Assume that

$$L(v_1) = p_1 v_1 - H(p_1)$$

$$L(v_2) = v_2 p_2 - H(p_2)$$

So

$$\frac{1}{2}L(v_1) + \frac{1}{2}L(v_2) = \frac{p_1 v_1 + p_2 v_2}{2} - \frac{1}{2}H(p_1) - \frac{1}{2}H(p_2) \qquad (15.1)$$

$$\begin{aligned} L(\frac{v_1+v_2}{2}) + \frac{1}{8\theta}|v_1-v_2|^2 &\geq \frac{v_1+v_2}{2}\frac{p_1+p_2}{2} - H(\frac{p_1+p_2}{2}) + \frac{1}{8\theta}|v_1-v_2|^2 \\ &\geq \frac{(v_1+v_2)\cdot(p_1+p_2)}{4} - \frac{1}{2}H(p_1) - \frac{1}{2}H(p_2) + \frac{\theta}{8}|p_1-p_2|^2 + \frac{1}{8\theta}|v_1-v_2|^2 \quad \text{by (36)} \\ &= \frac{1}{2}L(v_1) + \frac{1}{2}L(v_2) - \frac{(v_1-v_2)(p_1-p_2)}{4} + \frac{\theta}{8}|p_1-p_2|^2 + \frac{1}{8\theta}|v_1-v_2|^2 \quad \text{by (15.1)} \\ &\geq \frac{1}{2}L(v_1) + \frac{1}{2}L(v_2) - \frac{(v_1-v_2)\cdot(p_1-p_2)}{4} + \frac{|p_1-p_2||v_1-v_2|}{4} \qquad (15.2) \\ &\geq \frac{1}{2}L(v_1) + \frac{1}{2}L(v_2) \qquad \text{by Cauchy's inequality} \end{aligned}$$

Note that at (15.2), we used Arithmetic-Geometric Mean Inequality.

16. Assume that

$$u^1(x,t) = tL(\frac{x-y_1}{t}) + g^1(y_1)$$

$$u^2(x,t) = tL(\frac{x-y_2}{t}) + g^2(y_2)$$

Thus

$$\begin{aligned} u^1(x,t) - u^2(x,t) &= \min_{y\in\Re^n}\{tL(\frac{x-y}{t}) + g^1(y)\} - tL(\frac{x-y_2}{t}) + g^2(y_2) \qquad \text{put } y = y_2 \\ &\leq tL(\frac{x-y_2}{t}) + g^1(y_2) - tL(\frac{x-y_2}{t}) - g^2(y_2) \\ &= g^1(y_2) - g^2(y_2) \end{aligned}$$

Similarly

$$u^2(x,t) - u^1(x,t) \leq g^2(y_1) - g^1(y_1)$$

So either $|u^2(x,t) - u^1(x,t)| \leq |g^2(y_1) - g^1(y_1)|$ or $|u^1(x,t) - u^2(x,t)| \leq |g^1(y_2) - g^2(y_2)|$. In each case,

$$|u^1(x,t) - u^2(x,t)| \leq \sup_{\Re^n}|g^1 - g^2|$$

Bibliography

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