Problem Set 37
Circles Continued

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July 20, 2020

1 HW

- Solve AMC10 Mock test I sent in 75 mins.
- Solve section 2.
- Solve section 3.

2 Miscellaneous

1. (TJNMO-FR 2009) $ABCD$ is rectangle with $AB = 10, BC = 6$. The points E and F are chosen on the sides BC and CD respectively so that $CF = 5, CE = 2$. The lines AE and BF intersect at point G.

   i GE can be written as $\sqrt{\frac{a}{b}}$ where $(a, b) = 1$. What is $a + b$?

   ii GC can be written as $\sqrt{\frac{a}{b}}$ where $(a, b) = 1$. What is $a + b$?

   NOTE: This is a new problem

2. For how many integer pairs (a, b) with boundaries $1 \leq a \leq 17,\ 1 \leq b \leq 17$

   $$17 | 1 + 4a + 5b + 3ab?$$

3. (TJNMO-FR 2009) After x percent discount for a particular product, sales per day increased 50 percent and the income increased by 26 percent. Find x.

4. (TJNMO-FR 2009) Let AH be the altitude in the triangle ABC with $H \in BC$. Let K be the feet of the altitude from H to side AB. It is given that $AH = 6, AC = 10$ and $\angle HAC = 2\angle BAH$. The length of HK ca be represented as $\sqrt{\frac{m}{n}}$. What is $m + n$?

5. (TJNMO-FR 2009) Trapezoid $ABCD$ has $AB = 6, BC = 3, AB \parallel CD$. The angle bisector of $\angle ABC$ intersects with CD at E and $BE = 5$. What is AE^2?

   Additions

6. (TJNMO-FR 2009) For how many positive integer pairs (x, y) the equality

   $$xy^2 = 128(x - 1)^2$$

   holds?

7. (TJNMO-FR 2009) The Irvine-Los Angeles train is 120 meters long and it is moving 60km per hour towards LA. There is a bird at the back of the train moving at constant speed in the same direction as the train. It takes 21 second for the bird to go to the front of the train, and come back directly to the back without changing its speed (but it changes its direction when it reaches to the front.). What is the speed of the bird in terms of km per hour?

8. Let $f(x) = \frac{x^2+4x}{x^2-x-2}$ for $x \neq -1, 2$. Find the set as a union of intervals where $f(x) < 0$.

9. For real number $m > 1$, the function f defined as

$$f(x) = \frac{x^2 - mx + m - 1}{x^2 - m^2}$$

for all reals $x \neq \pm m$. The solution set of the inequality $f(x) < 0$ can be represented as a union of intervals $S = (a, b) \cup (c, d)$. if the sum of the length of the intervals is 15, what is m?

10. (TJNMO-FR 2009) Let M be the midpoint of the side AC of the triangle ABC. If $\angle DBC = 15, \angle BDC = 30$, find $\angle BAC$.
HINT: Choose point K on side BC so that $DK = DC$. This will lead lots of isosceles triangles.

11. Let point I be the intersection of angle bisectors of triangle ABC. The distance from I to side AB is 4. Outside of the circle, three isosceles triangles are constructed: $\triangle A_1BC, \triangle B_1AC, \triangle C_1AB$ with bases BC, CA, AB. Length of the height from A_1 to BC, B_1 to AC, C_1 to AB are all 2010.

12. **From Previous PS**

    (a) (TNMO 2018FR) ABC is right triangle with hypotenuse AB and it is given that $AC/BC = 3/4$. The interior circle touches sides BC and AC at D and E respectively. AD intersects with the incircle again at the point S. Similarly BE intersects with the incircle again at T. BE and AD intersect at point K.

    i Find AS/KD

    ii Find $(AS/TD)^2$

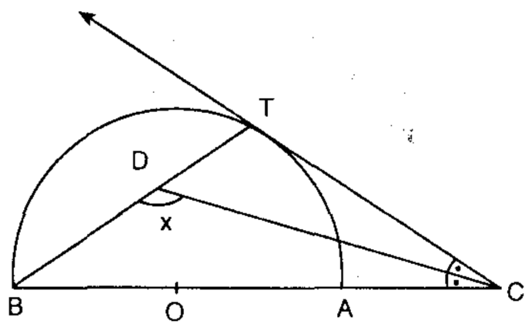
    (b) A positive integer $n \geq 2$ is called *fantastic* if there exist positive integer m such that for all $j \leq n-1$ divides m, but n is not divisor of m. In other words, n is the smallest non divisor of m. Find the number of *fantastic* numbers less than 27.

    (c) (HMMT-F 2014 Geometry 8) Let ABC be a triangle with sides $AB = 6$, $BC = 10$, and $CA = 8$. Let M and N be the midpoints of BA and BC, respectively. Choose the point Y on ray CM so that the circumcircle of triangle AMY is tangent to AN. Find the area of triangle NAY.

3 Circle Basics

1. **From Previous PS with hints**

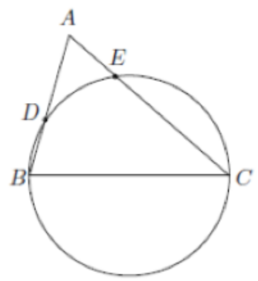
    i In the figure there is a half circle with center O and diameter AB, and CT is tangent to the semicircle at point T. If CD is angle bisector of $\angle BCT$, what is $\angle BDC = x$?



HINT: Start with $\angle TCD = \angle DCB = \alpha$ and chase angles.

ii (TJNMO-FR 2009) From a point A outside of the circle Γ the tangent AB is drawn, where B is the tangent point. Another line which passes through A cuts the circle Γ at points C and D. If $BC = 4, BD = 6$, what the maximum integer length that the segment AB can take?
HINT: Use similarity $\triangle ABC \sim \triangle ADB$. Find the ratios between AB, AC, AD. I.e., if $AB = 6x$, what are the lengths if AC and AD?. Then use triangle inequality in $\triangle BCD$ to create inequality for x.

iii (COMC 2011 B3) In the figure, BC is a diameter of the circle, where $BC = \sqrt{901}, BD = 1$, and $DA = 16$. If $EC = x$, what is the value of x?

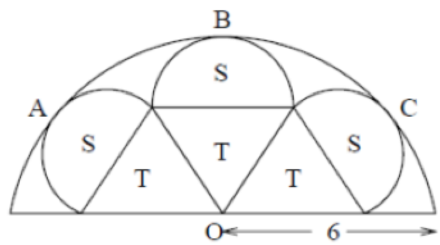


HINT: $\angle BDC = 90$ (Why?). So you know $CD = 30$. Then compute AC. Then use power of point.

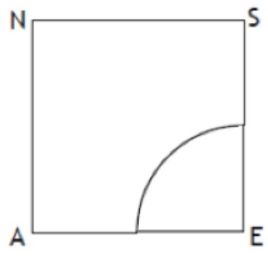
iv (AIME-II 2013 8) A hexagon that is inscribed in a circle has side lengths 22, 22, 20, 22, 22, and 20 in that order. The radius of the circle can be written as $p + \sqrt{q}$, where p and q are positive integers. Find $p + q$.
HINT: Draw the diameter that cuts the hexagon in half. Then mark the center. Try to apply Pythagorean's Thm.

2. (HMMT-F Geometry 2014 2-modified) Point P and line ℓ are such that the distance from P to ℓ is 24. Given that T is a point on ℓ such that $PT = 25$, find the radius of the circle passing through P and tangent to ℓ at T.

3. (COMC 1998 3) In the figure, each region T represents an equilateral triangle and each region S a semicircle. The complete figure is a semicircle of radius 6 with its centre O. The three smaller semicircles touch the large semicircle at points A, B and C . What is the radius of a semicircle S?

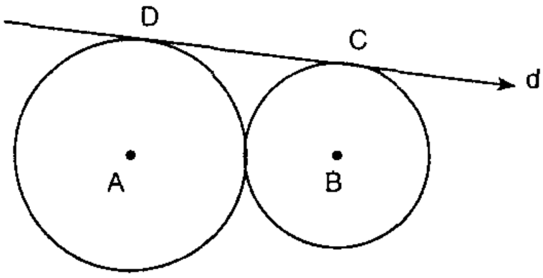


4. (HMMT Geometry 1998 5) Square SEAN has side length 2 and a quarter-circle of radius 1 around E is cut out. Find the radius of the largest circle that can be inscribed in the remaining figure.



5. (HMMT Guts 2001 10) Two concentric circles have radii r and $R > r$. Three new circles are drawn so that they are each tangent to the big two circles and tangent to the other two new circles. Find $\frac{R}{r}$.

6. In the figure two circles with centers A and B have common tangent line DC. The radius of the circle with center B is 4, and he radius of the circle with center A is 9. If two circles are tangent to each other at point E, what is the are of the triangle DCE?

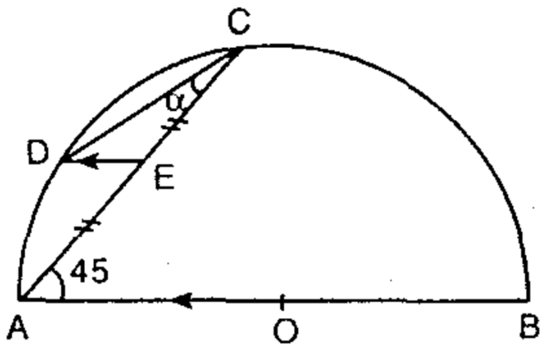


HINT: First compute DC (Do you recognize the figure.). Then Prove $\angle DEC = 90$. Then compute EC and ED using law of cos.

7. The circle with center O has diameter length $\sqrt{2425}$. Two chords AB and CD have midpoints M and N respectively. If $CD - AB = OM - ON = 2$, find the area of the triangle ODC.

Additions

8. In the figure point C is chose on the half circle with center O such that $\angle OAC = 45$. AB is the diameter and E is midpoint of AC. D is on the minor arc $\overset{\frown}{AC}$ such that $ED \parallel AB$. Find $\angle CDA$.



9. (AIME 1997 4) Circles of radii $5, 5, 8$, and $\frac{m}{n}$ are mutually externally tangent, where m and n are relatively prime positive integers. Find $m + n$.

10. (HMMT Geometry 2001 8) Point D is drawn on side BC of equilateral triangle ABC, and AD is extended past D to E such that angles EAC and EBC are equal. If BE = 5 and CE = 12, determine the length of AE.
HINT:Remember *Ptolemy's Theorem.* You may check it online if you forget it.