Problem Set 15
Area

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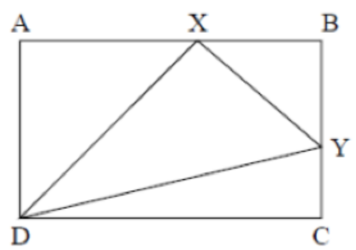
1 HW

1.1 Part I

- Solve AMC12-2014B in 90 mins. Check your answers and then continue solving the rest of the problems. Spend at least 10 mins on each problem. Then you may read the solutions from AOPS. Mark the ones you did not understand, we will go over them next class.
  NOTE:Let me know immediately if you did test recently and remember the problems. I would send another test in this case.
- Solve section 2
- Solve section 3
- **If you did not solve this part last week do this part also**
  From AOPS geometry book
  i Solve 8.64,67,68,70 at p242
  ii Read Ch9.
  iii Solve problems: 9.34,36,37,38,42,46,47,48 at p262
- (Proof Writing Practice Q2 with Hint)
  1. Now assume $\angle B > \angle C$ and $E \in BC$ so E, B, C align in that order. Let K be a points on line AC so that A is in the middle of K and C. It is given that $\angle KAE = \angle EAB$. In other words, AE is exterior angle bisector of $\hat{A}$. Prove that $EB/EC = AB/AC$.
     Hint: Apply law of sin to the triangle EAC.

2 Area

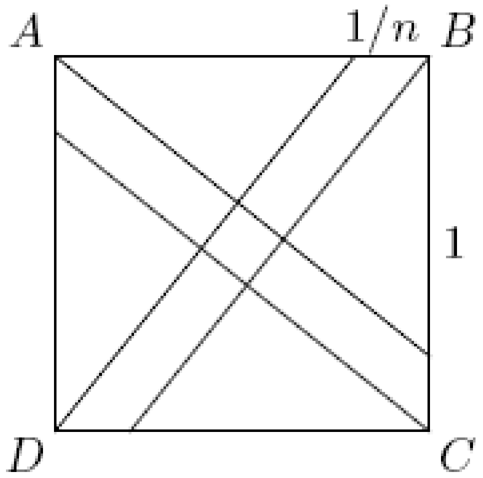
1. (COMC 1998 B2) ABCD is a rectangle and lines DX, DY and XY are drawn where X is on AB and Y is on BC. The area of triangle AXD is 5, the area of triangle BXY is 4 and the area of triangle CYD is 3. Determine the area of triangle DXY.



2. (Rice MT Team 2016 2) According to the Constitution of the Kingdom of Nepal, the shape of the flag is constructed as follows:

   Draw a line AB of the required length from left to right. From A draw a line AC perpendicular to AB making AC equal to AB plus one third AB. From AC mark off D making line AD equal to line AB. Join BD. From BD mark off E making BE equal to AB. Touching E draw a line FG, starting from the point F on line AC, parallel to AB to the right hand-side. Mark off FG equal to AB. Join CG. If the length of AB is 1 unit, what is the area of the flag?

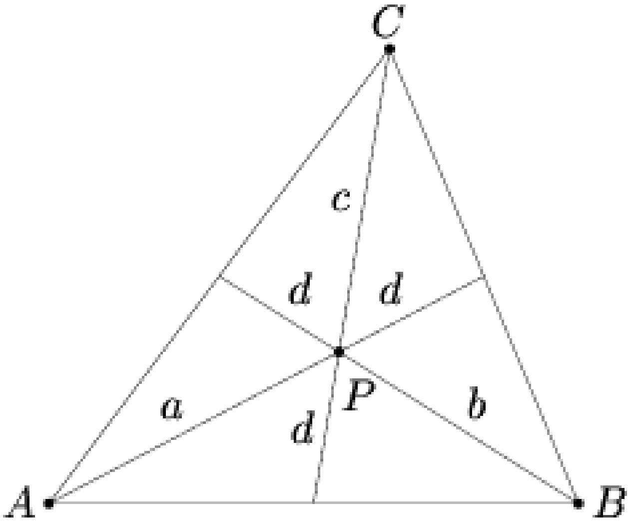
3. (AIME 1985 4) A small square is constructed inside a square of area 1 by dividing each side of the unit square into n equal parts, and then connecting the vertices to the division points closest to the opposite vertices. Find the value of n if the the area of the small square is exactly $\frac{1}{1985}$.



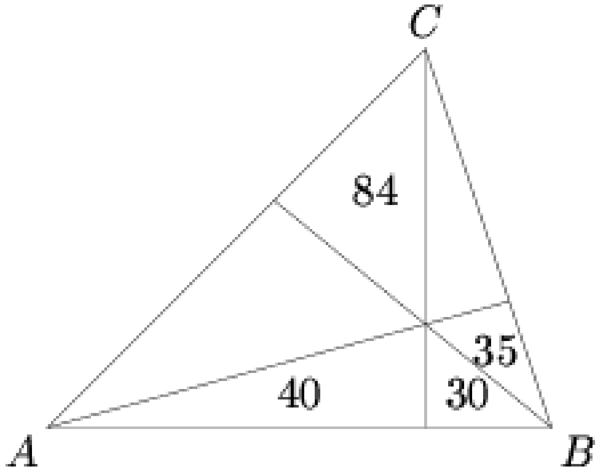
4. (HMMT-F 2014 Geometry 8) Let ABC be a triangle with sides $AB = 6$, $BC = 10$, and $CA = 8$. Let M and N be the midpoints of BA and BC, respectively. Choose the point Y on ray CM so that the circumcircle of triangle AMY is tangent to AN. Find the area of triangle NAY.

5. (Rice MT Geometry 2016 8) Natasha walks along a closed convex polygonal curve of length 2016. She carries a paintbrush of length 1 and walking all the way around paints all the area as far as she can reach on the outside of the curve. What is that area?

6. (AIME 1988 12) Let P be an interior point of triangle ABC and extend lines from the vertices through P to the opposite sides. Let a, b, c, and d denote the lengths of the segments indicated in the figure. Find the product abc if $a + b + c = 43$ and $d = 3$.



7. (AIME 1985 6) As shown in the figure, triangle ABC is divided into six smaller triangles by lines drawn from the vertices through a common interior point. The areas of four of these triangles are as indicated. Find the area of triangle ABC.



8. ABC is a triangle with sides $AB = 6,\ AC = 8,\ BC = 8$. M is the midpoint of the side BC. Let r_1 and r_2 be the inradii of the triangles ABM and ACM respectively. $\frac{r_1}{r_2}$ can be written as $\frac{a-\sqrt{b}}{c}$. Find $a + b + c$.

9. (AIME 1996 13) In triangle ABC, $AB = \sqrt{30}$, $AC = \sqrt{6}$, and $BC = \sqrt{15}$. There is a point D for which $\overline{AD}$ bisects $\overline{BC}$, and $\angle ADB$ is a right angle. The ratio $\frac{[ADB]}{[ABC]}$ can be written in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3 Miscellaneous

1. **From Previous PS with hints**

   i **Redo the calculations**

      i Find integers x and y so that $1234x + 4321y = 1$.
      Hint: Remember *Euclidean Algorithm*

      ii Find all integer pairs (x, y) so that $1234x + 4321y = 1$.

   ii (HMMT 2007 Guts) The equation $x^2 + 2x = i$ has two complex roots. Determine the product of their real parts.
   HINT: Complete into square, and write $i + 1 = re^{i\theta}$. Find length r and angle θ.

   iii (HMMT 2007 Guts) 6] A sequence consists of the digits $122333444455555\ldots$ such that the each positive integer n is repeated n times, in increasing order. Find the sum of the 4501st and 4052nd digits of this sequence.
   Assume m is 2 digit integer. Find the length of $122333444455555\ldots mmm\ldots m$

2. (HMMT Guts 2004) Find all positive integer solutions (m, n) to the following equation:

$$m^2 = 1! + 2! + \cdots + n!$$

   Use modular arithmetic

3. Find the last three digits of the number 2017^{2017}.

4. Let $P(x)$ be third degree polynomial with $P(-1) = -3, P(1) = 3$ and $P(3) = 9$. If 2 is a root of $P(x)$, find the reminder when $P(x)$ divided by $x - 4$.
   Remember how we solved a similar problem to this a week before.

5. (AOPS Mock AMC10 2017 Summer) The Tribonacci sequence is defined as $T_1 = 1, T_2 = 1, T_3 = 2$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n > 3$. What is the remainder when T_{2017} is divided by 12?

6. x and y are integers such that $23x + 12y = 1$.
   - Find the smallest value $|x| + |y|$ can take.
   - Find the second smallest value $|x| + |y|$ can take.
   - Find the third smallest value $|x| + |y|$ can take.
   - Is there a largest value of $|x| + |y|$?

7. (AOPS Mock AMC10 2017 Summer) Let $ABCD$ be a convex quadrilateral with $\angle ABD = 18, \angle ACB = 54, \angle ACD = 36, \angle ADB = 27$, and let P be the intersection of the two diagonals AC and BD. What is the degree value of $\angle APB$?
   HINT: Extend AC towards A and complete into cyclic quadrilateral.

8. (AOPS Mock AMC10 2017 Summer) Given that $\frac{5}{x+y} - \frac{1}{xy} = 1$, for positive reals x, y, the maximum value of x can be expressed in the form $\frac{a+\sqrt{b}}{c}$. What is the value of $a + b + c$?

9. (HMMT 2007 Guts) Determine the largest integer n such that $7^{2048} - 1$ is divisible by 2^n.

10. (HMMT 2007 Guts) Convex quadrilateral $ABCD$ has right angles $\angle A$ and $\angle C$ and is such that $AB = BC$ and $AD = CD$. The diagonals AC and BD intersect at point M. Points P and Q lie on the circumcircle of triangle AMB and segment CD, respectively, such that points P, M, and Q are collinear. Suppose that $\angle ABC = 160$ and $\angle QMC = 40$. Find $MP \cdot MQ$, given that $MC = 6$.

11. The triangle ABC has $AC = 9$. Choose point P on side AC so that $AP = BP = 4$. Assume that the circumcircle of triangle BPC is tangent to side AB. The angle bisector of $\angle BPC$ cuts the circumcircle of BPC at M. What is PM?

    Additions

12. (AHSME 1992) Let $i = \sqrt{-1}$. What is the product of the real parts of the roots of $z^2 - z = 5 - 5i$?

13. Let $f(x) = x^2 - ax + 2020$ where a is a real number. Find a if $f(2020) = f(1048)$.

14. Let $f(x) = x^3 - 4039x^2 + Nx + 1$ where N is an integer. Find the remainder of N when divided by 1000 if $f(2020) = f(2019)$.

15. Let $f(x) = x^2 - 1$ and $g(x) = x - 1$. Find the sum of integers n which does not satisfy
$$(f(g(n))) > g(n-1)$$

16. (AHSME 1975) If p, q and r are distinct roots of $x^3 - x^2 + x - 2 = 0$, then $p^3 + q^3 + r^3$ equals
    (A) -1 **(B)** 1 **(C)** 3 **(D)** 5 **(E)** none of these

17. Let p and q be real numbers. Find the minimal value of
$$(p^2 + 4\sqrt{2}p + 16)(q^2 - 3q + 4)$$

18. (COMC 1996 7) Triangle ABC is right angled at A. The circle with center A and radius AB cuts BC and AC internally at D and E respectively. If $BD = 20$ and $DC = 16$, determine AC^2.

19. For how many positive integers n less than 500
$$n^2 \equiv 6n + 66 \mod 75$$
    holds?

20. Let x_1, x_2, x_3 be the complex roots of the equation $x^3 - 2x^2 + 3x - 4 = 0$. Find the value of $x_1^3 + x_2^3 + x_3^3$.

21. Find the remainder when 7^{2048} divided by 250.

4 Day 2: Discrete Probability & Statistics I

4.1 Lecture Notes

Beginning Remark Probability is simply a tool to calculate/quantify the "likelihood" of that and "event" occur.

Notations & Theorem C2.1 Usually by E we denote the *Universal Space*. Here E can be either a discrete or continuum set. For any subset of A of E, we denote its *probability* by $p(A)$. For example, E can be modelled as the possible output of trowing a dice. $E = \{1, 2, \ldots, 6\}$ in this example. Assuming the dice is fair, then $p(i) = \frac{1}{6}$ for all $i = 1, 2, \ldots, 6$. Moreover $p(\text{output is even}) = p(\{2, 4, 6\}) = \frac{3}{6}$.

Here are some facts and corresponding examples.

- $p(E) = 1$ by definition.
- $p(A) + p(A^c) = 1$ where $A^c := E/A$
- If $A \cap B = \emptyset$ then $p(A \cup B) = p(A) + p(B)$
- $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
- (Conditional Probability) Probability of A given that B occurs:

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

- (Independent Events) We call events A and B are independent if $P(A) = p(A|B)$ or equivalently $p(A \cap B) = p(A) \cdot p(B)$. It means that the occurrence of the event B does not provide any information for the occurrence of event A, i.e. two events are *independent*.
- (Random Variable) Random variable is a function $\mathbb{X}E \to \mathbb{R}$ from universal set to reals. For example, we can define random variable for throwing a dice as the number that comes in the above face of the dice.
- (Expectation) $\mathbb{E}[\mathbb{X}] = \sum_{a \in E} p(\mathbb{X} = x) \cdot x$. For example, expected outcome of throwing a fair dice is $1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = 3.5$

Definition In a finite sequence $a_1, a_2, \ldots, a_n$, the *mod* is the number that appears the most, the *median* is the middle number when the sequence is ordered, and the *mean* is the arithmetic mean of $a_1, a_2, \ldots, a_n$.

Sample Problem I (HMMT Advanced 2001 6) There are two red, two black, two white, and a positive but unknown number of blue socks in a drawer. It is empirically determined that if two socks are taken from the drawer without replacement, the probability they are of the same color is $\frac{1}{5}$. How many blue socks are there in the drawer?

Solution Let the number of blue socks be $x > 0$. Then the probability of drawing a red sock from the drawer is $\frac{2}{6+x}$ and the probability of drawing a second red sock from the drawer is $\frac{1}{5+x}$, so the probability of drawing two red socks from the drawer without replacement is $\frac{2}{(6+x)(5+x)}$. This is the same as the probability of drawing two black socks from the drawer and the same as the probability of drawing two white socks from the drawer. The probability of drawing two blue socks from the drawer, similarly, is $\frac{x(x-1)}{(6+x)(5+x)}$. Thus the probability of drawing two socks of the same color is the sum of the probabilities of drawing two red, two black, two white, and two blue socks from the drawer:

$$3\frac{2}{(6+x)(5+x)} + \frac{x(x-1)}{(6+x)(5+x)} = \frac{1}{5}$$

Cross-multiplying and distributing gives $5x^2 - 5x + 30 = x^2 + 11x + 30$, so $4x^2 - 16x = 0$,and $x = 0$ or 4. But since $x > 0$,there are 4 blue socks.

4.1.1 Conditional Probability

1. Two dices are thrown.

   i Find the probability that one of the dice is 3.

   ii Now it is given that the sum of the numbers is 5. Find the probability that one of the dice came 3.

2. An urn contains 2 blue and 2 green balls. Two balls are selected at random without replacement, and you are told that one of the selected ball is blue. What is the probability that both of the selected balls are blue.

3. (AMC10 2011A) Two counterfeit coins of equal weight are mixed with 8 identical genuine coins. The weight of each of the counterfeit coins is different from the weight of each of the genuine coins. A pair of coins is selected at random without replacement from the 10 coins. A second pair is selected at random without replacement from the remaining 8 coins. The combined weight of the first pair is equal to the combined weight of the second pair. What is the probability that all 4 selected coins are genuine?

   (A) $\frac{7}{11}$ **(B)** $\frac{9}{13}$ **(C)** $\frac{11}{15}$ **(D)** $\frac{15}{19}$ **(E)** $\frac{15}{16}$

4.1.2 Basics

1. A company has 9 employees, 3 woman and 6 men. This company will send 3 randomly selected personal to holiday at Hawai. Find the probability that at least 1 men is selected?

2. Find the probability that a gambler gets at least 8 tails when he throws a fair coin 10 times.

3. The chair of space department at NASA will chose 5 monkeys randomly among 10 male and 6 female monkeys to sent them over a mission to Jupiter. Find the probability that 3 male and 2 female are chosen.

4. (HMMT 2005 General) In an election, there are two candidates, A and B, who each have 5 supporters. Each supporter, independent of other supporters, has a $\frac{1}{2}$ probability of voting for his or her candidate and a $\frac{1}{2}$ probability of being lazy and not voting. What is the probability of a tie (which includes the case in which no one votes)?

5. Alice has 3 boxes with different colors: blue, red and yellow. In the blue box there are 2 red and 3 yellow balls, in the red box there are 1 blue and 4 yellow balls, and in the yellow box there are 4 blue and 3 red balls. She chooses one box randomly then she takes one ball out from the chosen box. If the probability of choosing the blue box is twice as the probability of choosing the red box and triple as the probability of choosing the yellow box, find the probability that she chooses a blue ball.

6. 4 numbers are chosen from the set $\{1, 2, 3, \ldots, 10\}$. Find the probability that difference between the largest and the smallest chosen number is less than 6.

7. Two numbers are chosen from the set $\{1, 2, 3, \ldots 50\}$. Find the probability that one is twice of the other.

8. (HMMT 2005 Guts) What is the probability that in a randomly chosen arrangement of the numbers and letters in "HMMT2005," one can read either "HMMT" or "2005" from left to right? (For example, in "5HM0M20T," one can read "HMMT.")

9. (AMC10 2003B) A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out. What is the probability that all beads in the bag are red after three such replacements?

   (A) $\frac{1}{8}$ **(B)** $\frac{5}{32}$ **(C)** $\frac{9}{32}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{7}{16}$

4.1.3 Moderate

1. Neo and Trinity enters the Matrix simulation for training to hunt agents. When encounter and agent, both has 50 percent chance to kill the agent. If one can't kill the agent one dies. After dying they can reenter the Matrix for training. Throughout the training, Neo encounter 6 agents (including the infamous Agent Smith), and Trinity encounters 5 agents. Find the probability that Neo kills more agents than Trinity.

2. (TNMO FR 1997) A box consist of total 25 balls with colors white and red. Two different balls are chosen from the box. It is given that the probability of both balls are white is 0.54. Find the probability that both balls are red.

3. (HMMT 2007 Guts) A candy company makes 5 colors of jellybeans, which come in equal proportions. If I grab a random sample of 5 jellybeans, what is the probability that I get exactly 2 distinct colors?

4. (AMC10 2002A) Tina randomly selects two distinct numbers from the set 1, 2, 3, 4, 5, and Sergio randomly selects a number from the set 1, 2, ..., 10. What is the probability that Sergio's number is larger than the sum of the two numbers chosen by Tina?

   (A) 2/5 (B) 9/20 (C) 1/2 (D) 11/20 (E) 24/25

5. (AMC10A 2009) Three distinct vertices of a cube are chosen at random. What is the probability that the plane determined by these three vertices contains points inside the cube?

   (A) $\frac{1}{4}$ (B) $\frac{3}{8}$ (C) $\frac{4}{7}$ (D) $\frac{5}{7}$ (E) $\frac{3}{4}$

6. (AMC10A 2010) Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

7. (AMC10A 2010) Each of 2010 boxes in a line contains a single red marble, and for $1 \le k \le 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

   (A) 45 **(B)** 63 **(C)** 64 **(D)** 201 **(E)** 1005

8. (AMC10-2001) A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

   (A) $\dfrac{3}{10}$ **(B)** $\dfrac{2}{5}$ **(C)** $\dfrac{1}{2}$ **(D)** $\dfrac{3}{5}$ **(E)** $\dfrac{7}{10}$

9. (AMC12 2004A) Select numbers a and b between 0 and 1 independently and at random, and let c be their sum. Let A, B and C be the results when a, b and c, respectively, are rounded to the nearest integer. What is the probability that $A + B = C$?

   (A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) $\frac{3}{4}$

10. (HMMT-F 2014 Combinatorics 1) There are 100 students who want to sign up for the class Introduction to Acting. There are three class sections for Introduction to Acting, each of which will fit exactly 20 students. The 100 students, including Alex and Zhu, are put in a lottery, and 60 of them are randomly selected to fill up the classes. What is the probability that Alex and Zhu end up getting into the same section for the class?

11. (AIME 1990 9) A fair coin is to be tossed 10 times. Let i/j, in lowest terms, be the probability that heads never occur on consecutive tosses. Find $i + j$.

12. (HMMT Advanced 2002 7) A manufacturer of airplane parts makes a certain engine that has a probability p of failing on any given flight. Their are two planes that can be made with this sort of engine, one that has 3 engines and one that has 5. A plane crashes if more than half its engines fail. For what values of p do the two plane models have the same probability of crashing?

13. (HMMT Advanced 1999 3) An unfair coin has the property that when flipped four times, it has the same probability of turning up 2 heads and 2 tails (in any order) as 3 heads and 1 tail (in any order). What is the probability of getting a head in any one flip?

14. (AIME-I 2014 2) An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

15. (HMMT-F Guts 2014 4) Let D be the set of divisors of 100. Let Z be the set of integers between 1 and 100, inclusive. Mark chooses an element d of D and an element z of Z uniformly at random. What is the probability that d divides z?

4.1.4 Hard

1. (AMC10 2006A) A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

2. (HMMT 2005 Guts) If a, b, and c are random real numbers from 0 to 1, independently and uniformly chosen, what is the average (expected) value of the smallest of a, b, and c?

3. (AHSME 1992) An "unfair" coin has a 2/3 probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even?

   (A) $25\left(\frac{2}{3}\right)^{50}$ (B) $\frac{1}{2}\left(1-\frac{1}{3^{50}}\right)$ (C) $\frac{1}{2}$ (D) $\frac{1}{2}\left(1+\frac{1}{3^{50}}\right)$ (E) $\frac{2}{3}$

4. (AIME 1991 13) A drawer contains a mixture of red socks and blue socks, at most 1991 in all. It so happens that, when two socks are selected randomly without replacement, there is a probability of exactly $\frac{1}{2}$ that both are red or both are blue. What is the largest possible number of red socks in the drawer that is consistent with this data?

5. (HMMT Combinatorics 2003 7) You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled 1, 2, ... Each ball has probability $1/2^n$ of being put into box n. The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls?

6. (SMT Advanced-2015-04) Andy has two identical cups, the first one is full of water and the second one is empty. He pours half the water from the first cup into the second, then a third of the water in the second into first, then a fourth of the water from the first into the second and so on.Compute the fraction of the water in the first cup right before the 2015th transfer.