An Inequality Problem

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Abstract

I wrote this inequality problem in 2014 and it was published in the problem section of "The Mathematical Gazette, Volume 100, Issue 548" in July 2016. The problem is inspired from the problem Iran TST 2009, Problem 3.

Problem Let a, b, c, d be positive reals such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove

$$\sum_{cyc} \frac{2 + c(a-b)}{a+b+2c} \geq 2$$

Solution The statement is equivalent to

$$4 \leq \sum_{cyc} \frac{4 + 2c(a-b)}{a+b+2c} + (b-a) = \sum_{cyc} \frac{4 + b^2 - a^2}{a+b+2c} = \sum_{cyc} \frac{2a^2 + b^2 + c^2}{a+b+2c}$$

Lemma: For all x, y, z, t, p, q, r, s positive real numbers, the inequality

$$\frac{x^2}{p} + \frac{y^2}{q} + \frac{z^2}{r} + \frac{t^2}{s} \geq \frac{(x+y+z+t)^2}{p+q+r+s}$$

holds.

Proof: By Cauchy-Schwarz inequality

$$\underbrace{\left((\frac{x}{\sqrt{p}})^2 + (\frac{y}{\sqrt{q}})^2 + (\frac{z}{\sqrt{r}})^2 + (\frac{t}{\sqrt{s}})^2 \right)}_{LHS = \frac{x^2}{p} + \frac{y^2}{q} + \frac{z^2}{r} + \frac{t^2}{s}} \underbrace{((\sqrt{p})^2 + (\sqrt{q})^2 + (\sqrt{r})^2 + (\sqrt{s})^2)}_{p+q+r+s} \geq$$

$$\geq \left(\frac{x}{\sqrt{p}}\sqrt{p} + \frac{y}{\sqrt{q}}\sqrt{q} + \frac{z}{\sqrt{r}}\sqrt{r} + \frac{t}{\sqrt{s}}\sqrt{s} \right)^2 = (x+y+z+t)^2$$

which proves the Lemma. In our question, we take $x = \sqrt{2b^2 + c^2 + d^2}$ and $p = a + b + 2c$, and we choose y, z, t, q, r, s similarly. Applying the lemma,

$$\sum_{cyc} \frac{2a^2 + b^2 + c^2}{a+b+2c} \geq \frac{\left(\sum_{cyc} \sqrt{2a^2 + b^2 + c^2} \right)^2}{4(a+b+c+d)} = \frac{4(a^2 + b^2 + c^2 + d^2) + 2\sum \sqrt{2a^2 + b^2 + c^2}\sqrt{2b^2 + c^2 + d^2}}{4(a+b+c+d)} = (\star)$$

By Cauchy-Schwarz inequality,

$$\sqrt{2a^2 + b^2 + c^2}\sqrt{2b^2 + c^2 + d^2} = \sqrt{a^2 + a^2 + b^2 + c^2}\sqrt{b^2 + d^2 + b^2 + c^2} \geq ab + ad + b^2 + c^2$$

$$\sqrt{2b^2 + c^2 + d^2}\sqrt{2c^2 + d^2 + a^2} = \sqrt{b^2 + b^2 + c^2 + d^2}\sqrt{c^2 + a^2 + c^2 + d^2} \geq bc + ba + c^2 + d^2$$

$$\sqrt{2c^2 + d^2 + a^2}\sqrt{2d^2 + a^2 + b^2} = \sqrt{c^2 + c^2 + d^2 + a^2}\sqrt{d^2 + b^2 + d^2 + a^2} \geq cd + cb + d^2 + a^2$$

$$\sqrt{2d^2 + a^2 + b^2}\sqrt{2a^2 + b^2 + c^2} = \sqrt{d^2 + d^2 + a^2 + b^2}\sqrt{a^2 + c^2 + a^2 + b^2} \geq da + dc + a^2 + b^2$$

$$\sqrt{2a^2+b^2+c^2}\sqrt{2c^2+d^2+a^2} = \sqrt{a^2+b^2+a^2+c^2}\sqrt{c^2+d^2+a^2+c^2} \geq ac+bd+a^2+c^2$$
$$\sqrt{2b^2+c^2+d^2}\sqrt{2d^2+a^2+b^2} = \sqrt{b^2+c^2+b^2+d^2}\sqrt{d^2+a^2+b^2+d^2} \geq bd+ac+b^2+d^2$$

Adding them up results

$$\sum \sqrt{2a^2+b^2+c^2}\sqrt{2b^2+c^2+d^2} \geq 3(a^2+b^2+c^2+d^2)+2(ab+ac+ad+bc+bd+cd)$$

Thus

$$4(a^2+b^2+c^2+d^2)+2\sum \sqrt{2a^2+b^2+c^2}\sqrt{2b^2+c^2+d^2} \geq 10(a^2+b^2+c^2+d^2)+4(ab+ac+ad+bc+bd+cd) =$$

$$= 8(a^2+b^2+c^2+d^2)+2(a+b+c+d)^2 = 32+2(a+b+c+d)^2 \geq (AM-GM) \geq 16(a+b+c+d)$$

Therefore

$$(\star) \geq \frac{16(a+b+c+d)}{4(a+b+c+d)} = 4$$

which proves the inequality.